Classifying spaces and Q-construction

$\mathcal{C}$ small category (for Proj_R, take the "skeleton" to make it small)

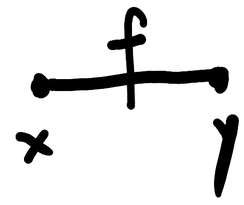
↓ nerve

$N\mathcal{C}$ simplicial set $\left(\cdots \to N_2 \rightrightarrows N_1 \rightrightarrows N_0 \right)$

N_2 = composable morphisms in $\mathcal{C}$; N_1 = $\mathrm{Mor}\,\mathcal{C}$; N_0 = $\mathrm{Obj}\,\mathcal{C}$

↓ geometric realization

$|N\mathcal{C}|$ top'l space (glue: $\cdot$ v $\quad v \in \mathrm{Obj}\,\mathcal{C}$



$f \in \mathrm{Mor}\,\mathcal{C}$



$\in N_2$

etc. in the natural way)

!!

$B\mathcal{C}$

(~~classifying space~~ of $\mathcal{C}$)

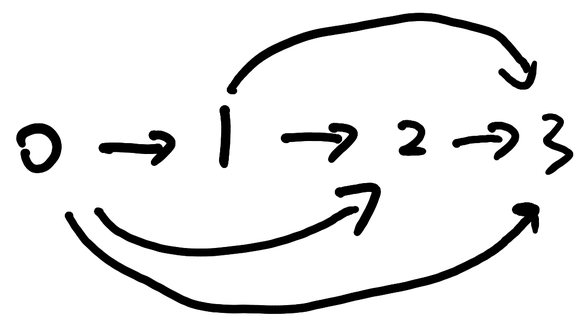
Rmk. We can thus talk about categories being contractible / homotopy equiv.

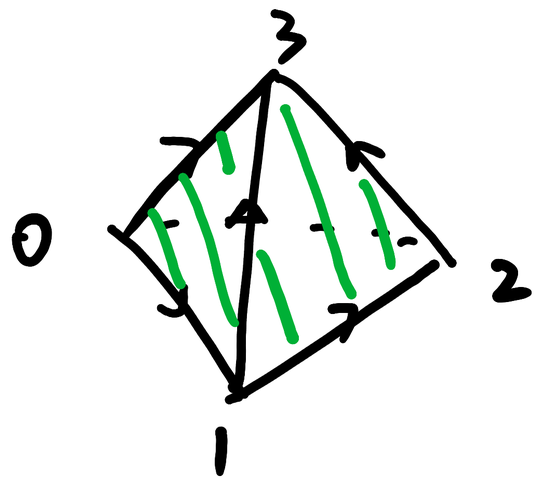
Ex 1) $\mathcal{C} = G$, $B\mathcal{C} = BG$

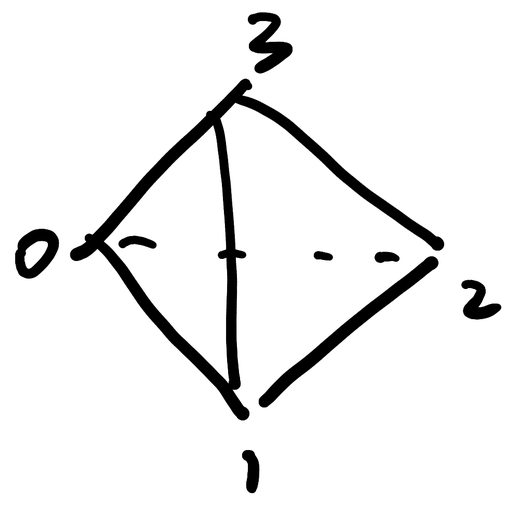
2) $\mathcal{C}$ = partially ordered set J

$B\mathcal{C}$ = simplicial complex for J

(in fact all simplicial complexes arise this way)

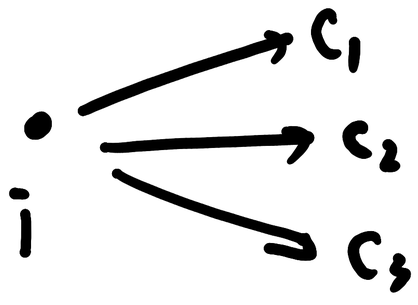
e.g. $J = \{0, 1, 2, 3\}$ $\quad \mathcal{C}:$ 

$B\mathcal{C} =$ 

Exercise. Find J s.t. $BJ =$ 

(not filled)

3) If $\mathcal{C}$ has a terminal / initial object, then $B\mathcal{C}$ is contractible



everything contracts to i

e.g. $\mathcal{C}$ = (skeleton of Proj_R)

0 is both initial and final

$\Rightarrow$ $B\mathcal{C}$ is contractible

$\Rightarrow$ $\pi_i(B\mathcal{C}) = 0$ does not give K-theory

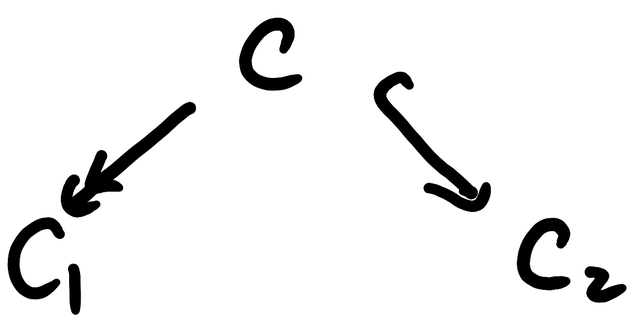
<u>Quillen's idea</u>. Modify $\mathcal{C}$ to $Q\mathcal{C}$ so that

$$\pi_1(BQ\mathcal{C}) = K_0(\mathcal{C})$$

Let $\mathcal{C}$ be an exact cat

Define $Q\mathcal{C}$ as the cat. where

- objects in $Q\mathcal{C}$ = obj in $\mathcal{C}$
- morphism $C_1 \to C_2$ given by diagrams



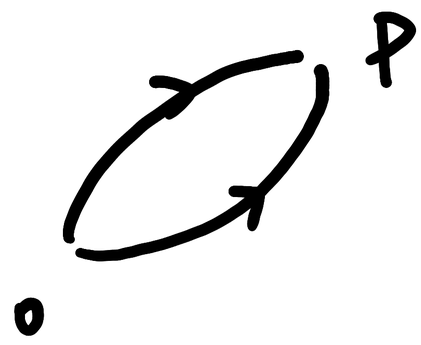
"invert the epimorphisms"

Idea.



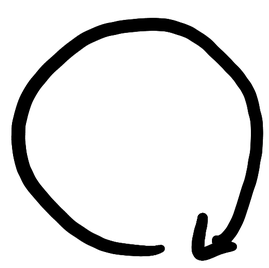
is contractible

If we "invert the map $P \to 0$",



is not contractible

and contracting along $0 \to P$ gives a loop

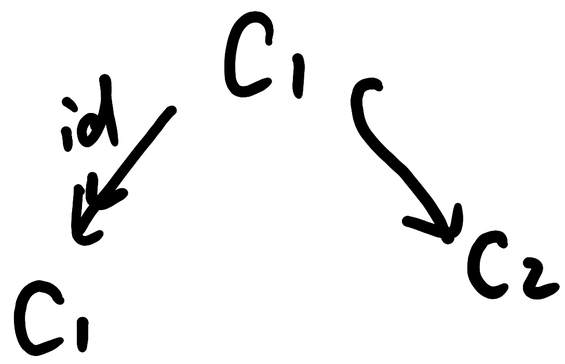


$0 = P$

loops in $\partial\mathcal{C}$ $\longleftrightarrow$ Obj in $\mathcal{C}$

$$\pi_1(\partial\mathcal{C}) \cong K_0(\mathcal{C})$$

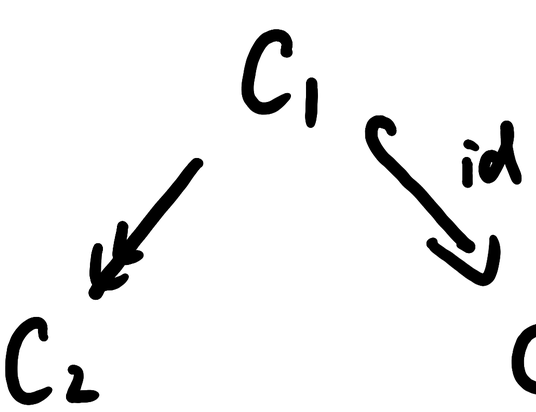
e.g. $C_1 \hookrightarrow C_2$ in $\mathcal{C}$ becomes



Denote this by $[C_1 \hookrightarrow C_2] \in Mor(\mathcal{QC})$

e.g. $C_1 \twoheadrightarrow C_2$ in $\mathcal{C}$ becomes

"a map from 0 to $\mathcal{C}$"



Denote this by $[C_1 \twoheadleftarrow C_2] \in Mor(\mathcal{QC})$

To compute $\pi_1(B\mathcal{QC})$, we may contract it along all $[0 \hookrightarrow C]$. $C \in Obj\,\mathcal{C}$

Then $\pi_1(B\mathcal{QC}) = \{\text{morphisms } [f] \text{ in } \mathcal{QC}\} / \sim$

$[f \circ g] = [f] \cdot [g]$

$[0 \hookrightarrow C] = [id]$

$\forall\, C \in Obj\,\mathcal{C}$

(as all objs in $\mathcal{C}$ are identified)

Exercise. 1) All morphisms in $\mathcal{QC}$ is a composition

$$[C_1 \twoheadleftarrow C_2][C_2 \hookrightarrow C_3]$$

2) All $[C_2 \hookrightarrow C_3]$ is contracted

(Hint: Check $[0 \hookrightarrow C_2][C_2 \hookrightarrow C_3] = [0 \hookrightarrow C_3]$)

3) All $[C_1 \twoheadrightarrow C_2] = [0 \twoheadleftarrow K]$.

where $0 \to K \to C_1 \to C_2 \to 0$

Exercise.

$$\pi_1(B\mathcal{QC}) = \frac{\{\text{morphisms } [f] \text{ in } \mathcal{QC}\}}{[f \circ g] = [f] \cdot [g]}$$

$[0 \twoheadleftarrow C]$

$\downarrow$

$[C]$

$\cong K_0(\mathcal{C})$

$[0 \hookrightarrow C] = [\mathrm{id}]$

$\forall C \in \mathrm{Obj}$

$\downarrow$

Grothendieck relation in $K_0(\mathcal{C})$

The map $K_0(\mathcal{C}) \longrightarrow \pi_1(B\mathcal{QC})$

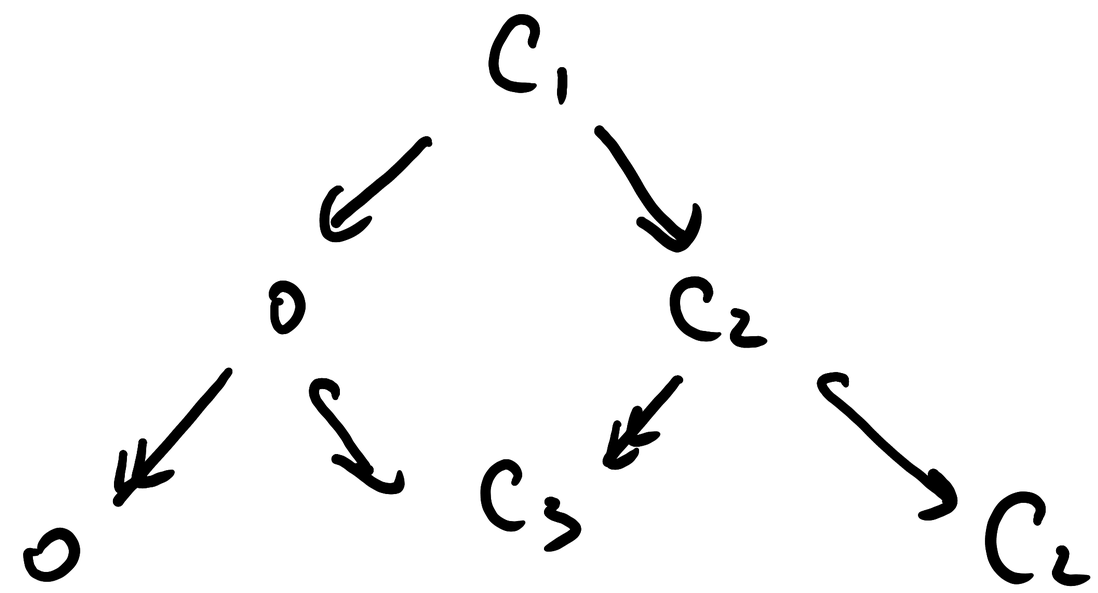
$$C \longmapsto [0 \hookleftarrow C] \quad \text{is surj.}$$

by description of morphisms in $\mathcal{QC}$;

The Grothendieck relation translates to natural product structure in $\pi_1(B\mathcal{QC})$:

$$0 \to C_1 \to C_2 \to C_3 \to 0 \quad \text{exact}$$

$\Rightarrow \exists$ diagram



$$\Rightarrow \quad [0 \hookrightarrow C_3]\,[C_3 \hookleftarrow C_2]$$

$$= [0 \hookleftarrow C_1]\,[C_1 \hookrightarrow C_2]$$

$$\text{so} \quad [0 \hookleftarrow C_2] = [0 \hookleftarrow C_3] \cdot [C_3 \hookleftarrow C_2]$$

$$= [0 \hookleftarrow C_3] \cdot [0 \hookleftarrow C_1]$$

Define $K_i\mathcal{M} := \pi_{i+1}(B\mathcal{QM}, 0)$

[Technical notes:

① This is indep. of the choice of zero object in $\mathcal{M}$

② This def^{n} works for an exact cat of $\mathcal{M}$ whose isom. classes of obj & morphisms form a set]

Properties of K-groups

① An exact functor $f: \mathcal{M} \to \mathcal{M}'$ induces a map $K_i\mathcal{M} \to K_i\mathcal{M}'$

② $K_i(\mathcal{M} \times \mathcal{M}') \xrightarrow{\sim} K_i\mathcal{M} \oplus K_i\mathcal{M}'$

(since $\mathcal{Q}(\mathcal{M} \times \mathcal{M}') \simeq \mathcal{QM} \times \mathcal{QM}'$)

③ The functor $\oplus : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ induces

$\oplus_* : K_i(\mathcal{M} \times \mathcal{M}) \xrightarrow{\sim} K_i\mathcal{M} \oplus K_i\mathcal{M} \to \mathcal{M}$

This coincide w/ the sum in $K_i\mathcal{M}$ as an

belian group

④ K_i commutes w/ filtered colimit:

$$K_i(\varinjlim M_j) = \varinjlim K_i(M_j)$$

Define $K_i(R) = K_i(\mathrm{Proj}_R)$

$G_i(R) = K_i(\mathrm{Coh}_R)$

Oct 14

Some clarifications:

① Def^n of exact cat & composition of morphisms in $\mathcal{QC}$

② $G_i(R) := K_i($ coherent R-mod $)$

↑ not f.g. R-mod, as it need not be abelian for R non-Noetherian