Timeline.

1967 Milnor defined K_2

1968 Bass introduced K_{-n}

1970 - 1971 Milnor developed K_n^M

1972 Quillen introduced higher K-theory

Recall.

R Noetherian

$\Rightarrow$ ① $G_i(R) \xrightarrow{\sim} G_i(R[t])$, $\forall i$

② $G_i(R[t, t^{-1}]) \cong G_i(R) \oplus G_{i-1}(R)$, $\forall i$

For k-theory, Bass defines the **NK-groups**

$$NK_i(R) := \operatorname{coker}\left(K_i(R) \to K_i(R[t]) \right)$$

$\forall i$

$\exists$ splitting induced by

$R \to R[t] \xrightarrow{t=0} R$

We have $K_i(R[t]) \cong K_i(R) \oplus NK_i(R)$

e.g. $R = k[\varepsilon]/(\varepsilon^2)$, $1 + \varepsilon t \in R[t]^\times \subseteq K_1(R[t])$

is a nonzero elem in $NK_1(R[t])$

Fundamental thm of K-theory (when n=1)

$x \mapsto (x, x)$

$$0 \to K_1(R) \xrightarrow{\Delta} K_1(R[t]) \oplus K_1(R[t^{-1}])$$

$$\to K_1(R[t,t^{-1}]) \xrightarrow{\partial} K_0(R) \to 0$$

$(a,b) \mapsto a-b$

$\exists$ splitting s

$P \mapsto$ (multiplication by t on $P[t,t^{-1}]$)

What is ∂?

For $g \in GL_n(R[t,t^{-1}])$, choose N large s.t. $A(t) = t^N g(t,t^{-1}) \in M_n(R[t])$

Then $\partial g = [\operatorname{coker} A(t)] - N\cdot[R^n]$

$R[t]^n \xrightarrow{A[t]} R[t]^n$

Exercise. ① $\partial t = [R]$, $\partial t^m = m[R]$

② $\partial\left(P[t,t^{-1}] \xrightarrow{t} P[t,t^{-1}]\right) = [P]$ so $\partial s = id$

This implies the classical thm of Bass-Heller-Swan:

$$K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R) \oplus NK_1(R)^{\oplus 2}$$

(by diagram chasing:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
0 \to & K_1(R) & \xrightarrow{\Delta} & K_1(R) \oplus K_1(R) & \longrightarrow & K_1(R) & \to 0 \\
 & & \downarrow & & & & \\
0 \to & K_1(R) & \xrightarrow{\Delta} & K_1(R[t]) \oplus K_1(R[t^{-1}]) & \to & K_1(R[t,t^{-1}]) \to K_0(R) & \to 0 \\
 & & \downarrow & & & & \\
 & & NK(R)^{\oplus 2} & & & &)
\end{array}$$

Bass: Define $K_{-1}(R) := \operatorname{coker}\left(K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t,t^{-1}]) \right)$

Then we get Bass-Heller-Swan in one lower degree:

$$K_0(R[t,t^{-1}]) \cong K_0(R) \oplus K_1(R) \oplus NK_0(R)^{\oplus 2}$$

Pf. $K_1(R[s,t,t^{-1}]) \oplus K_1(R[s^{-1},t,t^{-1}]) \to K_1(R[s,s^{-1},t,t^{-1}]) \to K_0(R[t,t^{-1}]) \to 0$

$\cong$ $\cong$

$K_1(R[s]) \oplus K_1(R[s^{-1}]) \to K_1(R[s,s^{-1}]) \to K_0(R) \to 0$

$\oplus K_0(R[s]) \oplus K_0(R[s^{-1}]) \to K_0(R[s,s^{-1}]) \to K_{-1}(R) \to 0$

$\oplus NK_1(R[s]) \oplus NK_1(R[s^{-1}]) \to NK_0(R[s,s^{-1}]) \to NK_0(R) \to 0$

$$K_0(R[t,t^{-1}]) \cong K_0(R) \oplus K_{-1}(R) \oplus NK_0(R)^{\oplus 2}$$

Philosophy: Any result relating K_1 and K_0 extend to negative K-theory

Mayer - Victoris

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & S/I \end{array}$$

Cartesian square of rings

$$\begin{array}{ccccc} \vdots & & \updownarrow & & \\ \cdots = \operatorname{Spec} S/I & \longrightarrow & Y = \operatorname{Spec} S & = & \diagup \\ \downarrow & & \downarrow & & \\ \cdots = Z & \hookrightarrow & X = \operatorname{Spec} R & = & \alpha \\ \| & & & & \\ \operatorname{Spec} R/I & & & & \end{array}$$

Then Milnor's patching thm gives an exact seq

$$K_1(R) \to K_1(S) \oplus K_1(R/I) \to K_1(S/I)$$

$$\to K_0(R) \to K_0(S) \oplus K_0(R/I) \to K_0(S/I)$$

Can we extend this to the right by negative k-theory ?

Yes! To show

Fundamental thm for K_0

$$K_0(R) \to K_0(S) \oplus K_0(R/I) \to K_0(S/I)$$

$$\to K_{-1}(R) \to K_{-1}(S) \oplus K_{-1}(R/I) \to K_{-1}(S/I) \to 0$$

$$\cdots \to K_0(S[t,t^{-1}]) \oplus K_0(R/I[t,t^{-1}]) \to K_0(S/I[t,t^{-1}])$$

$$\to K_0(S[t]) \oplus K_0(S[t^{-1}]) \oplus K_0(R/I[t]) \oplus K_0(R/I[t^{-1}]) \to K_0(S/I[t]) \oplus K_0(S/I[t^{-1}])$$

Definition of K_{-1}

long exact seq relating K_1, K_0

Define $K_{-n}(R) := \operatorname{coker}\left(K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \to K_{-n+1}(R[t,t^{-1}]) \right)$

This extends various exact seq. for K_1 and K_0 to the right.

Next time: Compute $K_{-1}(\mathbb{Z})$

Oct 26

Last time. $K_{-n}(R) := \operatorname{coker}\big(K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \to K_{-n+1}(R[t,t^{-1}]) \big)$

For scheme X, define

$$K_{-n}(X) := \operatorname{coker}\big(K_{-n+1}(X \times \mathbb{A}^1) \oplus K_{-n+1}(X \times \mathbb{A}^1) \to K_{-n+1}(X \times \mathbb{G}_m) \big)$$

By functoriality, all long exact seq. relating K_0, K_1 extend to the right by negative K-theory

e.g. Milnor square

$$\begin{array}{ccc} R & \to & S \\ \downarrow & & \downarrow \\ R/I & \to & S/I \end{array}$$

i.e. $S \otimes_R R/I \cong S/I$

$$\Rightarrow \quad K_1(R) \to K_1(S) \oplus K_1(R/I) \to K_1(S/I)$$

$$\to K_0(R) \to K_0(S) \oplus K_0(R/I) \to K_0(S/I)$$

$$\to K_{-1}(R) \to K_{-1}(S) \oplus K_{-1}(R/I) \to \cdots$$

Rmk. ① Milnor's patching thm

$$\Phi : Proj(R) \to \underline{Proj(S) \times_{Proj S/I} Proj(R/I)}$$

$$P \mapsto (P \otimes_R S,\ P/IP,\ can)$$

cat. with objs

(P_1, P_2, α) — $P_1/IP_1 \xrightarrow{\alpha} S/I \otimes_{R/I} P_2$

P_1: S-mod, P_2: R/I-mod

gives a six term MV seq.

$$K_2(R) \to K_2(S) \oplus K_2(R/I) \to \cdots$$

(see Weibel's K-book for proof)

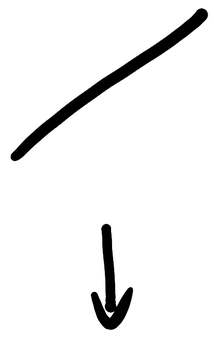
② Does not extend to K_3 are beyond

(see Swan's "Excision in alg. K-theory")

Ex/ $R = k[x,y]/(y^2 - x^2(x+1))$

$S = k[t]$

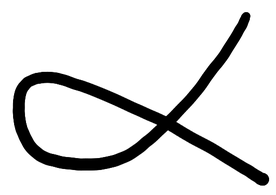
$I = (t) = (x,y)$

$\operatorname{Spec} S/I =$  $\to$  $= \operatorname{Spec} k[t]$

t



$(t^2-1,\ t(t^2-1))$

$\operatorname{Spec} R/I =$ $\cdot \to$  $= \operatorname{Spec} R$

$\rightsquigarrow \quad K_0(\alpha) \to K_0(k[t]) \oplus K_0(k) \to K_0(k \oplus k)$

$(a,\ b) \quad \mapsto (a-b,\ a-b)$

$\to K_{-1}(\alpha) \to K_{-1}(/ \amalg \cdots) = 0$

$\Rightarrow K_{-1}(\alpha) = \operatorname{coker}(\mathbb{Z}^2 \to \mathbb{Z}^2) \cong \mathbb{Z}$

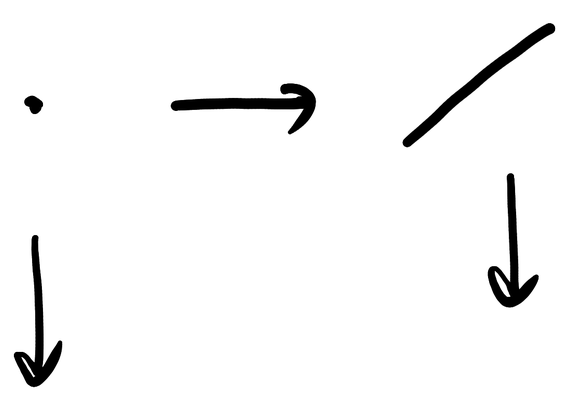
$(a,b) \mapsto (a-b,\ a-b)$

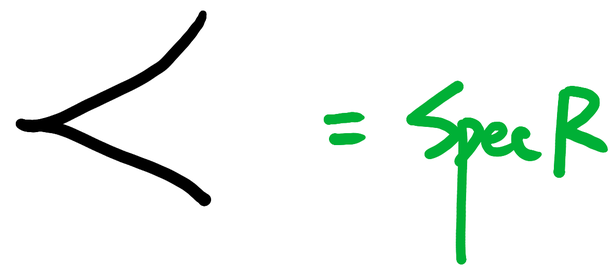
Ex/ $R = k[x,y]/(y^2 - x^3)$

$S = k[t]$

$I = (x,y) \subset R$

$= (t) \subset S$

$\mathrm{Spec}\, S/I =$  $= \mathrm{Spec}\, k[t]$

$\mathrm{Spec}\, R/I =$  $= \mathrm{Spec}\, R$

$$\rightsquigarrow \quad K_0(<) \to K_0(k[t]) \oplus K_0(k) \to K_0(k)$$

$$\to K_{-1}(<) \to K_{-1}(/ \amalg \cdot) = 0$$

$$\Rightarrow K_{-1}(<) = \mathrm{Coker}(\mathbb{Z}^2 \to \mathbb{Z}) = 0$$

$(a,b) \mapsto a-b$

Exercise. Show $K_{-1}($  $) = \mathbb{Z}^2$

$K_{-1}(X) = 0$

In fact, for a connected curve C,

$$K_{-1}(C) = \mathbb{Z}^N \quad \text{where}$$

$$N = \sum_{p \in C_{sing}} (\underbrace{r_p}_{\#\text{branches of } C \text{ at } p} - 1) - \underbrace{c}_{\#\text{ irred cpn}} + 1$$

Rmk. If C is a (reduced) connected complex alg curve, then $W_0 H^1(C, \mathbb{Q}) = \mathbb{Q}^N$ is given by the same formula!

↑ weight filtration on the singular coh of C

(This hints on some connections between Hodge theory and the negative K-theory)

What happens in higher dim ?

① For surfaces, the Milnor square

$$\begin{array}{ccc} E & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

X' = normalization (need not be smooth)

Z = points or (possibly singular) curve

$\rightsquigarrow$ MV seq. not so helpful in studying K_{-*}

For blowup square

$$\begin{array}{ccc} E & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

$\rightsquigarrow$ MV seq. need not hold for K-theory

Weibel: For resolution $\hat{\tilde{X}} \to X$ of normal surface X, factor into

$$\tilde{X} \longrightarrow X' \longrightarrow X$$

normalization; blowup along lci locus

both parts induce MV on K-theory

But this factorization does not generalize to higher dim ...

② In general, two ways to get MV seq:

1) Kerz - Strunk - Tame :

$$\begin{array}{ccc} E & \to & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \to & X \end{array}$$

abstract blowup square

i.e. pullback square

& $\tilde{X} - E \to X - Z$

Then $\cdots \to K_i(X) \to$ "$\lim_n$" $K_i(Z_n) \oplus K_i(\tilde{X})$

$\to$ "$\lim_n$" $K_i(E_n) \to \cdots$

Great for theoretical applications (e.g. Weibel's conj.)

Dificult to compute "$\lim_n$"

2) Weibel : Introduce "homotopy K-theory"

KH

Then for any abstract blow up square

$$\begin{array}{ccc} E & \to & \tilde{X} \\ \downarrow & & \downarrow \\ Z & \to & X \end{array} \quad \rightsquigarrow \quad \cdots \to KH_i(X) \to KH_i(\tilde{X}) \oplus KH_i(Z) \to KH_i(E) \to \cdots \quad \forall i$$

We will follow approach 2) to study negative K-theory: (*)

- Use (*) to study KH ← related to weight filtration
- Use K-regularity to relate K to KH

↑ related to Hodge filtration