Mixed Hodge complex of smooth quasi-projective varieties

Let X be a smooth, quasi-projective complex variety. We saw in class that each rational cohomology group $H^n(X,\mathbb{Q})$ carries an increasing weight filtration $W_\bullet$, and its complexification $H^n(X,\mathbb{C}) = H^n(X,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$ carries a decreasing Hodge filtration $F^\bullet$, such that each $\mathrm{Gr}^W_k H^n(X,\mathbb{Q})$ carries a pure Hodge structure of weight k, with Hodge filtration induced by $F^\bullet$.

In this note, we explain how to define the weight filtration on the level of rational complexes, and why it agrees with the weight filtration given by logarithmic forms after complexification.

Set up

Let $j\colon X \hookrightarrow \overline{X}$ be a smooth projective compactification, with $D = \overline{X}\setminus X$ a simple normal crossings divisor.

- On $K_{\mathbb{Q}} = Rj_*\mathbb{Q}_X$, we have a weight filtration defined by truncation
$$W_m Rj_*\mathbb{Q}_X = \tau_{\le m} Rj_*\mathbb{Q}_X,$$
where
$$(\tau_{\le m}K)^q = \begin{cases} K^q, & q<m,\\ \ker(d\colon K^m \to K^{m+1}), & q=m,\\ 0, & q>m.\end{cases}$$
- On $K_{\mathbb{C}} = \Omega^\bullet_{\overline{X}}(\log D)$, we have a weight filtration defined by
$$\begin{aligned} W_m\Omega^q_{\overline{X}}(\log D) &= \sum_{i=0}^{m} \Omega^{q-i}_{\overline{X}}\wedge\Omega^i_{\overline{X}}(\log D)\\ &= \{\text{logarithmic } q\text{-forms with } \le m \text{ denominators}\}.\end{aligned}$$

Question

Why is there a filtered quasi-isomorphism
$$(Rj_*\mathbb{Q}_X,\tau)\otimes_{\mathbb{Q}}\mathbb{C} \simeq (\Omega^\bullet_{\overline{X}}(\log D), W)?$$

Proof. First, the logarithmic Poincaré lemma gives quasi-isomorphisms in the derived category
$$Rj_*\mathbb{Q}_X\otimes_{\mathbb{Q}}\mathbb{C} \simeq Rj_*\mathbb{C}_X \simeq \Omega^\bullet_{\overline{X}}(\log D).$$
We need to check:

(i) These comparisons respect the filtrations, i.e. we have a map
$$(\tau_{\le m}Rj_*\mathbb{Q}_X)\otimes_{\mathbb{Q}}\mathbb{C} \longrightarrow W_m\Omega^\bullet_{\overline{X}}(\log D).$$
Indeed, this map is obtained by composing $(\tau_{\le m}Rj_*\mathbb{Q}_X)\otimes_{\mathbb{Q}}\mathbb{C} \simeq \tau_{\le m}Rj_*\mathbb{C}_X \cong \tau_{\le m}\Omega^\bullet_{\overline{X}}(\log D)$ with the natural map $\tau_{\le m}\Omega^\bullet_{\overline{X}}(\log D) \to W_m\Omega^\bullet_{\overline{X}}(\log D)$, defined by the vertical maps in the following diagram:
$$\begin{array}{ccccccccccc}
\tau_{\le m}\Omega^\bullet_{\overline{X}}(\log D): & \mathcal{O}_{\overline{X}} & \to & \Omega^1_{\overline{X}}(\log D) & \to\cdots\to & \Omega^{m-1}_{\overline{X}}(\log D) & \to & \ker d^m & \to & 0 & \to\cdots\\
& \| & & \| & & \| & & \downarrow & & \downarrow & \\
W_m\Omega^\bullet_{\overline{X}}(\log D): & \mathcal{O}_{\overline{X}} & \to & \Omega^1_{\overline{X}}(\log D) & \to\cdots\to & \Omega^{m-1}_{\overline{X}}(\log D) & \to & \Omega^m_{\overline{X}}(\log D) & \to & W_m\Omega^{m+1}_{\overline{X}}(\log D) & \to\cdots
\end{array}$$

(ii) They induce quasi-isomorphisms on the associated graded complexes, i.e.

$$(\mathrm{Gr}_m^\tau Rj_*\mathbb{Q}_X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathrm{Gr}_m^W \Omega^\bullet_{\overline{X}}(\log D).$$

Indeed, as we saw in class, computation using the local topology of a simple normal crossings divisor gives

$$\begin{aligned}(\mathrm{Gr}_m^\tau Rj_*\mathbb{Q}_X) \otimes_{\mathbb{Q}} \mathbb{C} &\simeq (R^m j_*\mathbb{Q}_X \otimes_{\mathbb{Q}} \mathbb{C})[-m] \\ &\simeq (a_m)_*\Omega^\bullet_{D^{[m]}}[-m],\end{aligned}$$

and the residue maps give

$$\mathrm{Gr}_m^W \Omega^\bullet_{\overline{X}}(\log D) \cong (a_m)_*\Omega^\bullet_{D^{[m]}}[-m],$$

where $D^{[m]}$ is the disjoint union of the m-fold intersections of components of D, and $a_m \colon D^{[m]} \to \overline{X}$ is the natural map.

Putting these together gives the quasi-isomorphism

$$(\tau_{\leq m} Rj_*\mathbb{Q}_X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} W_m\Omega^\bullet_{\overline{X}}(\log D). \qquad \square$$

References:

- P. Deligne, *Théorie de Hodge II*, Proposition (3.1.8), p. 33.
- P. Deligne, *Théorie de Hodge III*, Scholie (8.1.8), p. 30.