Additivity thm

$0 \to F' \to F \to F'' \to 0$ exact seq. of functors

$M \to M'$

Then $F_* = F'_* + F''_*$ as group hom

$K_i M \to K_i M'$

Applications

① $K_i(X)$ is a $K_0(X)$-module

By def^{n}, $K_i(X) := K_i(\underbrace{P(X)}_{\text{cat of v.b. on } X})$

Then $- \otimes_{\mathcal{O}_X} \underbrace{E}_{\text{v.b. on } X} : P(X) \to P(X)$ induces

a $K_0(X)$-action on $K_i(X)$

This is well-def bc if $0 \to E_1 \to E_2 \to E_3 \to 0$

in $P(X)$,

$$\Rightarrow\ 0 \to \otimes E_1 \to \otimes E_2 \to \otimes E_3 \to 0$$

is an exact seq. of functors

$$\Rightarrow\ [\otimes E_2] = [\otimes E_1] + [\otimes E_3] \quad \text{on } K_i(X)$$

② (K-theory of graded modules)

$A = \bigoplus_{n\geq 0} A_n$ graded ring

$$\Rightarrow K_i\left(\begin{array}{c}\text{cat. of}\\ A\text{-mod.}\end{array}^{\text{proj.}}\ \text{graded}\right) \cong K_i(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$$

Pf idea. For a graded A-mod $P = \bigoplus_{n\in\mathbb{Z}} P_n$,

$F_n P :=$ A-mod generated by P_k, $k \leq n$

Then $P \mapsto F_n P / F_{n-1} P \cong P_n \otimes_{A_0} A(-n)$

are exact functors on the cat. of graded A-mod

By additivity thm (and K_i commutes w/ filtered colimit)

$$[\mathrm{id}] = \sum_{n\in\mathbb{Z}} [F_n/F_{n-1}] \quad \text{on } K_i\left(\begin{array}{c}\text{graded}\\ A\text{-mod}\end{array}\right)$$

image generates $K_i(A_0)\, t^n$

This gives the isomorphism

$$K_i(\text{graded } A\text{-mod}) \cong K_i(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$$

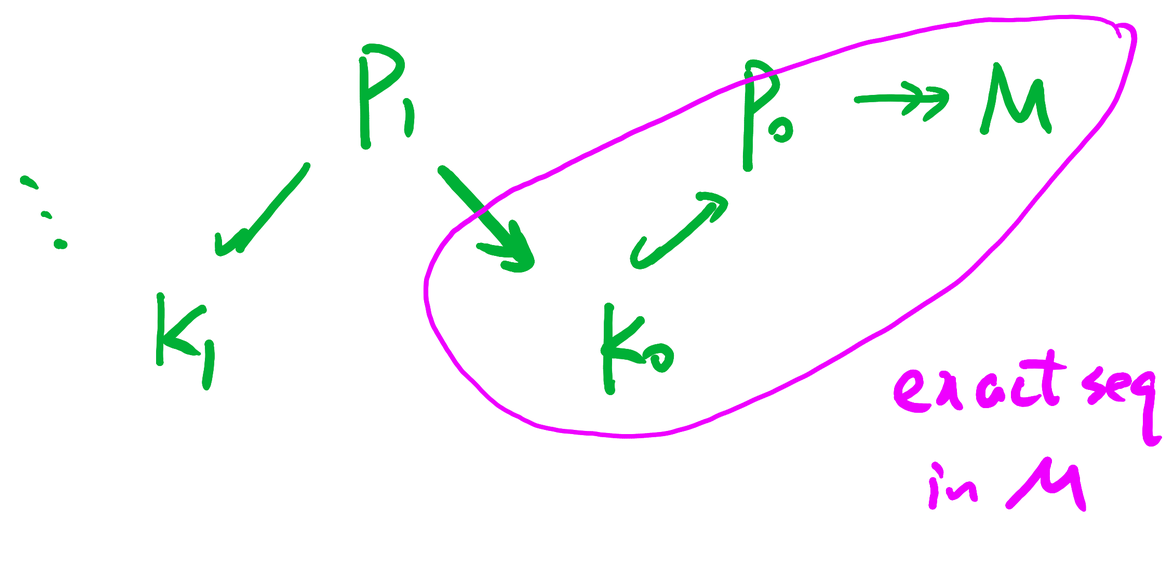
$$P \longmapsto \sum_{n \in \mathbb{Z}} \left[F_n / F_{n-1}\right] P$$

$$= \sum_{n \in \mathbb{Z}} P_n \, t^n$$

Resolution thm

$P \subseteq M$ full subcat

If any obj $M \in \mathcal{M}$ has a resolution by obj in P, then $K_i(P) \xrightarrow{\sim} K_i(\mathcal{M})$ is isom, $\forall i$



Applications

R regular ring

$$\Rightarrow K_i(R) \cong G_i(R)$$

$$\| \qquad\qquad \|$$

$$K_i(\text{proj } R\text{-mod}) \qquad K_i(\text{coh. } R\text{-mod})$$

(Any f.g. A-mod is resolved by proj A-mod)

Devissage and localization in abelian categories

A = abelian cat (where isom. classes are sets)

$B \subseteq A$ nonempty full subcat. closed under subobj, quotients & finite products

(hence B is abelian, $B \hookrightarrow A$ exact)

Not enough $B \hookrightarrow A$ is abelian

e.g. $B = \mathbb{Q}$-v.s. $\subseteq A$ = ab. gps

Thm (Devissage)

Suppose every object M in A has a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ s.t. M_j/M_{j-1} are in B, $\forall j$.

Then $QB \to QA$ is a homotopy equiv.

(In part, $K_i B \xrightarrow{\sim} K_i A$)

Cor. $I \subset R$ nilpotent ideal

$$\Rightarrow G_i(R/I) \cong G_i(R)$$

Pf. Take $A =$ coherent R-mod

$B =$ coh. R/I-mod

$\forall$ R-mod M, $0 = I^n M \subseteq I^{n-1} M \subseteq \cdots \subseteq M$

$\uparrow$ for n large

quotients are R/I-mod. $\square$.

Rmk.
This does not work for K-theory
(proj R-modules are not closed under quotients)

e.g. $K_i(\mathbb{Z}/p) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}/q^{n-1}, & i \text{ odd} \\ 0, & i \text{ even} \end{cases}$

$K_i(\mathbb{Z}/p^2\mathbb{Z})$ complicated

"On the K-theory of $\mathbb{Z}/p^n$"
Antieau - Krause - Nikolaus

Thm (Localization)

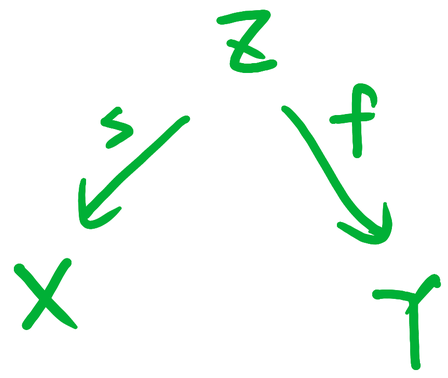
Let $B \subseteq A$ be a Serre subcat

closed under subobject & quotient obj

A/B be the quotient cat.

obj = obj of A

morphism =



S is a B-isom i.e. kers, cokers $\in B$

Then there is a long exact seq

$$\cdots \to K_i(B) \to K_i(A) \to K_i(A/B) \to K_{i-1}(B) \to \cdots$$

Cor. R Dedekind domain, $F = \mathrm{Frac}(R)$

$$\rightsquigarrow \to K_{i+1}(F) \to \bigoplus_{\mathfrak{m}} K_i(R/\mathfrak{m}) \to K_i(R) \to K_i(F) \to \cdots$$

[Take A = f.g. R-mod
B = torsion R-mod
A/B = f.g. F-mod]

Corrections : ① $K_i\left(\begin{smallmatrix}\text{f.g. graded}\\ \text{proj } S\text{- mod}\end{smallmatrix}\right) \xrightarrow{\sim} K_i(S_0) \otimes_{\mathbb{Z}} \mathbb{Z}[t,t^{-1}]$

where $S = \bigoplus_{n\geq 0} S_n$ is graded

Recall: Isom is given by

$$M \longmapsto \sum_{n\in\mathbb{Z}} \left(F_nM/F_{n-1}M\right) t^n$$

This is well-def on K_i bc F_n/F_{n-1} is exact on the cat of proj. modules

② $K_i\left(\begin{smallmatrix}\text{f.g. graded}\\ S\text{- mod}\end{smallmatrix}\right) \xrightarrow{\sim} G_i(S_0) \otimes_{\mathbb{Z}} \mathbb{Z}[t,t^{-1}]$

if ① S is positively graded

② S is flat over S_0

③ $S_0 = S/S_+$ has finite Tor-dim over S

Isom is given by

$$M \longmapsto \sum_{n\in\mathbb{Z}} \left(F_nM/F_{n-1}M\right) t^n$$

whenever F_n/F_{n-1} is exact

Homotopy invariance

R Noetherian ring $\Rightarrow$ $G_i(R) \cong G_i(R[t])$, $\forall i$

In part., R regular $\Rightarrow$ $K_i(R) \cong K_i(R[t])$, $\forall i$

Pf. Let $S = \bigoplus_{n \geq 0} \left(F_n R[t] \right) \cdot z^n$ be the

Rees ring of $R[t]$.

Apply localization thm to

B = f.g. graded S-mod where z acts nilpotently

$\cong$ f.g. graded mod over $S/zS \cong R[t]$
devissage

A = f.g. graded S-mod

A/B = f.g. graded S-mod where z acts invertibly

$\cong$ f.g. graded $S[z^{-1}]$-module

$\cong R[t][z, z^{-1}]$

$\cong$ f.g. $R[t]$-mod

(since z shifts the grading, a graded $R[t][z,z^{-1}]$-mod is determined by an $R[t]$-mod in deg 0)

By localization, have LES

$$
\begin{array}{ccc}
\vdots & & \circlearrowright \\
\downarrow & & \downarrow \\
K_i(\text{f.g. graded } R[t]\text{-mod}) & \longrightarrow & G_i(R)[t,t^{-1}] \\
\downarrow & & \downarrow \text{Claim: This map identifies as } 1-t \\
K_i(\text{f.g. graded } S\text{-mod}) & \xrightarrow{\simeq} & G_i(R)[t,t^{-1}] \\
\downarrow & & \downarrow \\
K_i(\text{f.g. } R[t]\text{-mod}) & \xrightarrow{\simeq} & G_i(R[t]) \\
\downarrow & & \downarrow \\
\vdots & & 0
\end{array}
$$

Assuming the claim, we have SES

$$0 \to G_i(R)[t,t^{-1}] \xrightarrow{1-t} G_i(R)[t,t^{-1}] \to G_i(R[t]) \to 0$$

$$\Rightarrow \quad G_i(R[t]) \cong \operatorname{Coker}(1-t) \cong G_i(R) .$$

<u>For the claim</u> :

$$\begin{array}{ccc} G_i(R)[t,t^{-1}] & \xrightarrow{\varphi} & G_i(R)[t,t^{-1}] \\ \downarrow M & & \uparrow 4 \\ M \otimes_R S/\mathbb{Z}S & \longmapsto & M \otimes_R S/\mathbb{Z}S \\ \downarrow & & \uparrow \\ K_i(\text{f.g. graded } R[t]\text{-mod}) & \longrightarrow & K_i(\text{f.g. graded } S\text{-mod}) \\ \downarrow S/\mathbb{Z}S & & \end{array}$$

Resolve $M \otimes_R S/\mathbb{Z}S$ (not Tor-indep from $S_0 = R/\text{induced from } R$)

by $\quad M \otimes_R S(-1) \xrightarrow{\mathbb{Z}} M \otimes_R S$

$F_0 = M \otimes_R S \qquad F_1 = M \otimes_R S$

so $\quad \varphi(M) = \psi(M \otimes_R S) - \psi(M \otimes_R S(-1))$

$$= \sum_{n \in \mathbb{Z}} F_n(M \otimes_R S) / F_{n-1}(M \otimes_R S) \cdot t^n$$

$$- \quad \cdots \qquad = [M] - [M]t$$

Rmk. This fails for K-theory

(the proof doesn't work bc Proj_R is not an abelian cat)

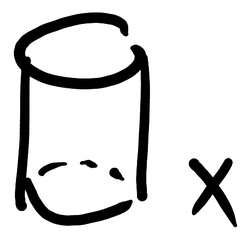
e.g. $R = \mathbb{C}[x,y,z]/(x^2+y^3+z^7)$

Bloch-Murthy $\Rightarrow$ $K_0(R) \neq K_0(R[t])$

Oct 19

Last time: $G_i(R) \cong G_i(R[t])$, $\forall i$

$\mathrm{Spec}\, k[t] = \mathbb{A}^1 \hookleftarrow I = [0,1]$

$\mathrm{Spec}\, R[t] = X \times \mathbb{A}^1 \hookleftarrow$  X

Expects G-theory of  X
= G-theory of X

$\mathrm{Spec}\, k[t,t^{-1}] = \mathbb{G}_m \hookleftarrow S^1$

$\mathrm{Spec}\, R[t,t^{-1}] = X \times \mathbb{G}_m \hookleftarrow X \times S^1$

Expect G-theory of $X \times \mathbb{G}_m$
to contain two copies of $G_*(X)$

Fundamental thm of G-theory

R Noetherian $\Rightarrow$ $G_i(R[t,t^{-1}]) \cong G_i(R)$

$\oplus\, G_{i-1}(R)$, $\forall i$

Pf. Apply localization thm to

$\mathcal{B}$ = coh. $R[t]$-mod where t acts nilpotently

$\rightsquigarrow$ $K_i(\mathcal{B}) \cong K_i(\text{f.g. } R\text{-mod})$

$\uparrow$ devissage

$= G_i(R)$

$\mathcal{A}$ = coh. $R[t]$-mod

$\rightsquigarrow$ $K_i(\mathcal{A}) \cong G_i(R[t])$

$\mathcal{A}/\mathcal{B}$ = coh. $R[t]$-mod where t acts invertibly

$\cong$ coh. $R[t,t^{-1}]$-mod

$\rightsquigarrow$ $K_i(\mathcal{A}/\mathcal{B}) \cong G_i(R[t,t^{-1}])$

We have LES

$$\cdots \to G_i(R) \xrightarrow{0} G_i(R[t]) \to G_i(R[t,t^{-1}])$$

($\exists$ splitting)

$$\to G_{i-1}(R) \xrightarrow{0} G_{i-1}(R[t])$$

Claim: The maps $G_*(R) \to G_*(R[t])$ are zero

induced by $M \mapsto M$

R-mod $\to$ $R[t]$-mod where t acts trivially

Pf. By additivity thm, $0 \to M[t] \xrightarrow{t} M[t] \to M \to 0$

(This is a SES of functors

{coherent R-mod} $\to$ {coherent $R[t]$-mod})

gives $(M \mapsto M[t])_* + (M \mapsto M)_*$

$= (M \mapsto M[t])_*$ on G_*

$\Rightarrow (M \mapsto M)_* = 0$ on G_* $\square$

Thus the LES breaks up into SES

$$0 \to G_i(R[t]) \to G_i(R[t,t^{-1}]) \to G_i(R) \to 0$$

Exercise: Check there is a splitting given by

$$G_i(R[t,t^{-1}]) \xrightarrow{t=1} G_i(R) \xrightarrow{\sim} G_i(R[t])$$

$$\Rightarrow G_i(R[t,t^{-1}]) \cong G_i(R[t]) \oplus G_i(R), \ \forall i$$

□.

Rmk. (Fundamental thm of K-theory)

R any ring. There is an exact seq

$$0 \to K_n(R) \to K_n(R[t]) \oplus K_n(R[t^{-1}]) \to K_n(R[t,t^{-1}])$$

$$\to K_{n-1}(R) \to 0$$

To prove this, one needs to extend the localization thm to "K-theory of v.b."
(work of Quillen, Thomason-Trobaugh ...)

See Weibel's K-book for details

$n=0$ $\quad \cdots \to K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t,t^{-1}])$

$$\to K_{-1}(R) \rightsquigarrow$$ leads to Bass' def^{n} of negative K-theory

On the proof of additivity thm / resolution / ...

Key input : Quillen's thm A

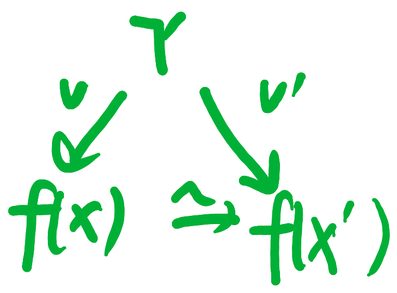
Let $f: C \to C'$ be a functor between cat C and C' s.t.

$\forall\, \gamma \in \mathrm{obj}\, C'$, $\gamma \downarrow f$ is contractible

comma cat with
- obj = $(x, v) : \gamma \xrightarrow{v} f(x)$
- morphism : $x \to x'$

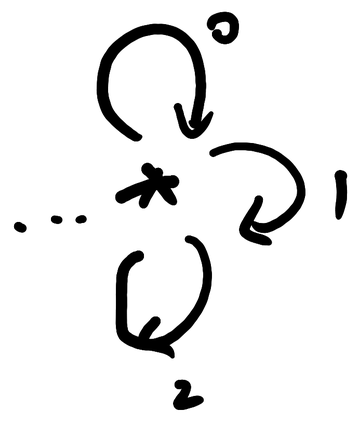
Then f is a homotopy equivalence (i.e. f induces $BC \xrightarrow{\simeq} BC'$)

s.t.



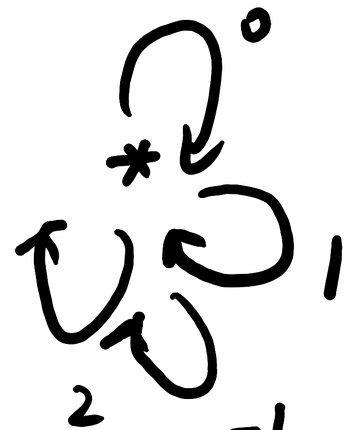
Ex/ $C = \mathbb{N}$, $C' = \mathbb{Z}$

$B\mathbb{N}$



$\simeq S^1$

$B\mathbb{Z} =$



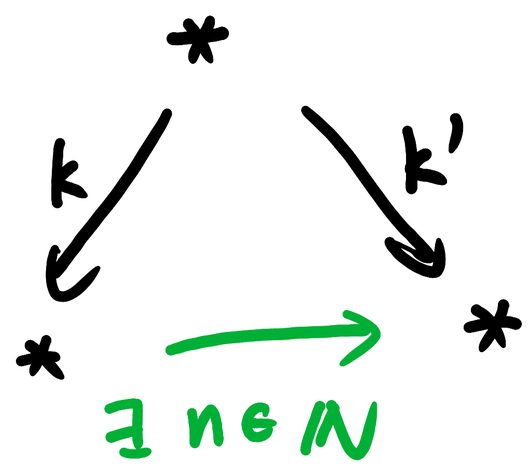
$\simeq S^1$

To see $B\mathbb{N} \hookrightarrow B\mathbb{Z}$ is a homotopy equiv. using thm A:

For the only object $* \in \mathbb{Z}$,

$*\downarrow f$ has objects : $* \xrightarrow{k} *$, $k \in \mathbb{Z}$

morphism



i.e. when $k \leq k'$

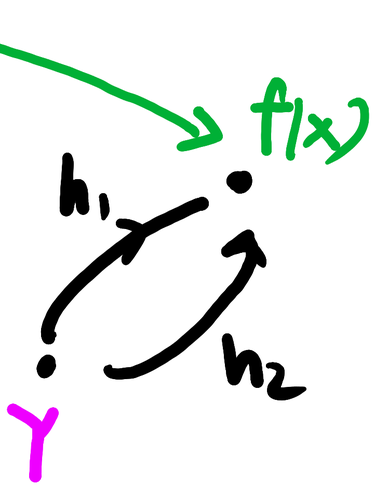
so $*\downarrow f \simeq (\mathbb{Z}, \leq)$ is contractible

$\overset{\text{thm A}}{\Rightarrow}$ $B\mathbb{N} \hookrightarrow B\mathbb{Z}$ is homotopy equiv

Ex/ $C =$



$C' =$



$\Upsilon \downarrow f = h_1 \amalg h_2$ is not contractible

$BC \not\simeq BC'$

Additivity thm:

$0 \to F' \to F \to F'' \to 0$ SEs of functors $\mathcal{M} \to \mathcal{M}'$

$\rightsquigarrow \quad F_* = F'_* + F''_*$ as gp hom $K_i\mathcal{M} \to K_i\mathcal{M}'$

Strategy: Reduce to split SES by showing

$\mathcal{M} \times \mathcal{M} \longrightarrow \Sigma$ is homotopy equiv. by thm A

$(M_1, M_2) \mapsto [0 \to M_1 \to M_1 \oplus M_2 \to 0]$

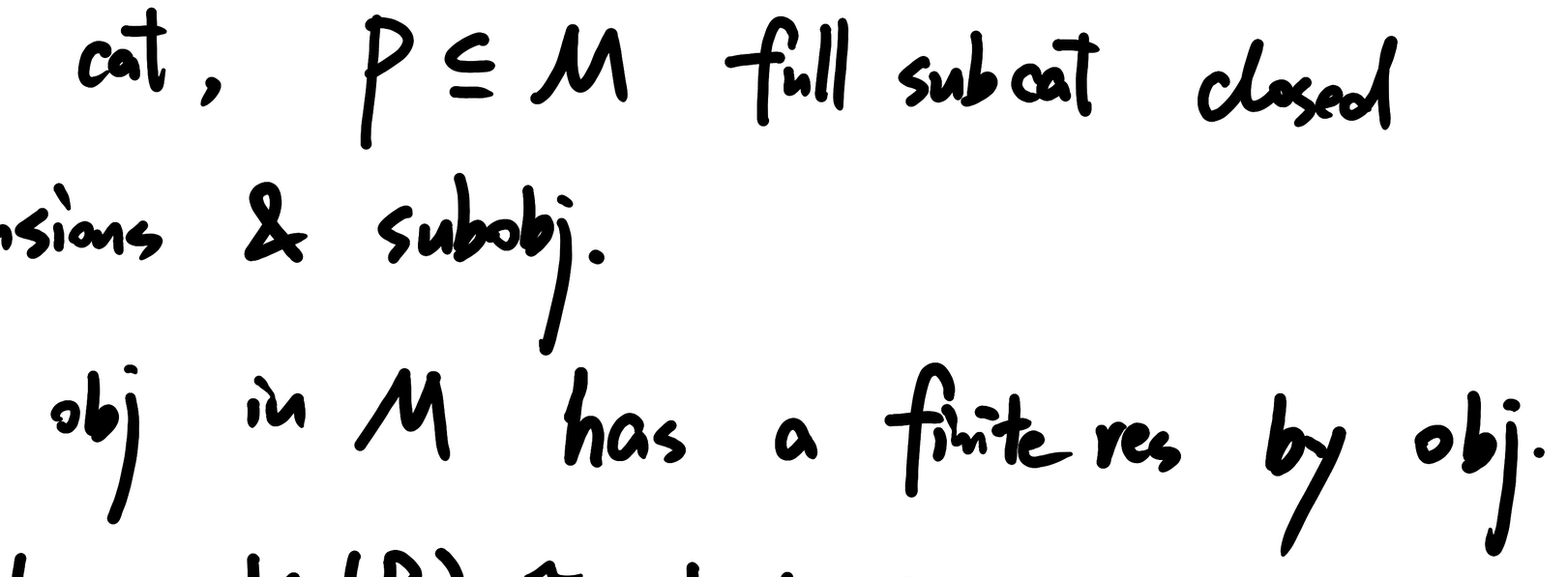
cat of SES

Resolution thm

$\mathcal{M}$ exact cat, $\mathcal{P} \subseteq \mathcal{M}$ full subcat closed under extensions & subobj.

If every obj in $\mathcal{M}$ has a finite res by obj. in $\mathcal{P}$, then $K_i(\mathcal{P}) \xrightarrow{\sim} K_i(\mathcal{M})$

Strategy: Factor $Q\mathcal{P} \xrightarrow{g} C \xrightarrow{f} Q\mathcal{M}$



By Quillen's thm A.

P is closed under subobj $\rightsquigarrow$ g is hom. equiv

P - - extension $\rightsquigarrow$ f is hom. equiv

$\Rightarrow$ $QP \xrightarrow{f \circ g} QM$ is hom. equiv

$\Rightarrow$ $K_i P \xrightarrow{\sim} K_i M$