Last time.

Def^n (Milnor). $K_2(R) = \ker(St(R) \to E(R))$

- $St(R)$ is generated by $e_{ij}(r)$ / "obvious" relations

so $K_2(R)$ = "extra relations in $E(R)$"

- Steinberg showed

$$0 \to K_2(R) \to St(R) \to E(R) \to 0$$

is a universal central extension of $E(R)$

general theory $\Rightarrow$ $K_2(R) = H_2(E(R), \mathbb{Z})$

Historical timeline

1950 - early 1960s : K_0, K_1 become established

1962 : Steinberg's work in group theory

~1967 : Milnor became interested and defined $K_2 = \ker(St \to E)$

Ex/ $K_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \langle w^4 \rangle$

where $w = e_{12}(1)\, e_{21}(-1)\, e_{12}(1)$

(In $E(R)$, w evaluates to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

so $w^4 = 1$ in $E(R)$

But w^4 cannot be expressed in terms of Steinberg relations)

Ex/ $K_2(k[t_1, \cdots, t_n]) = K_2(k)$ ($\mathbb{A}^1$-inv)

$K_2(k[t, t^{-1}]) = K_2(k) \oplus K_1(k) = K_2(k) \oplus k^\times$

(Bass - Heller - Swan)

Thm (Matsumoto).

k field

$$\Rightarrow K_2(k) \cong k^\times \otimes_{\mathbb{Z}} k^\times / \langle a \otimes (1-a) : a \neq 0, 1 \rangle$$

Image of $a \otimes b$ in $K_2(k)$ is denoted $\{a, b\}$

Ex/ • $K_2(\text{finite field}) = 0$

e.g. $K_2(\mathbb{F}_3) = 0$ bc $\mathbb{F}_3$ is gen by -1

and $\{-1, -1\} = 0$

• $K_2(\mathbb{Q}) \overset{\text{Tate}}{=} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{p \text{ odd prime}} (\mathbb{Z}/p\mathbb{Z})^\times$

• $K_2(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \oplus$ uniquely divisible

• $K_2(\mathbb{C})$ uniquely divisible

(i.e. $\forall\, n \in \mathbb{Z}^+,\ a \in K_2$

$nb = a$ for some $b \in K_2$)

Historical timeline

1968 Matsumoto's thesis describes $K_2(F)$

1970 Milnor introduces $K_*^M(F)$

Matsumoto's work motivates

$$K_n^M(k) = (k^\times)^{\otimes n} \Big/ \langle a_1 \otimes \cdots \otimes a_n : a_i + a_{i+1} = 1 \rangle$$

<u>Milnor's conjecture / Bloch-Kato</u>

We have isom.

$$K_n^M(k)/\ell\, K_n(k) \xrightarrow{\sim} H^n_{\text{ét}}(\mathrm{Spec}\, k, \mu_\ell^{\otimes n})$$

$\forall\, n,\ \ell \geq 0$

motivates Voevodsky to introduce motivic coh.

Rmk. Milnor K-theory $K_n^M(k)$ $=$ / $\neq$ Quillen K-theory $K_n(k)$ $n=0,1,2$ in general

<u>Exercise.</u> $K_n^M(\text{finite field}) = 0, \quad \forall\, n \geq 2$

(But $K_n(\text{finite field}) \neq 0$ for n odd)