The Whitehead group and K_1

K_0 : object (vector bundles / proj. R-mod)

K_1 : morphism $\quad \leftrightarrow$ invertible maps $R^n \to R^n$ or $GL_n(R)$

Def^n. $K_1(R) := GL(R)/E(R)$

where $GL(R) = \varinjlim_n GL_n(R)$

$$\left(\begin{array}{ccc} GL_n(R) & \longrightarrow & GL_{n+1}(R) \\ A & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right)$$

stabilization

$$E(R) = \varinjlim_n E_n(R)$$

where $E_n(R)$ is the $n \times n$ elementary matrices gen. by $e_{ij}(r)$, $r \in R$

$$\begin{pmatrix} 1 & & & \\ & \ddots & r & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \leftarrow (i,j)\text{-th entry}$$

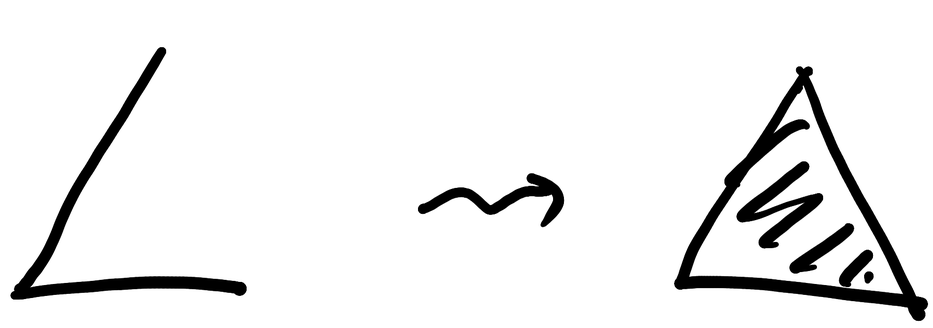
e.g. $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \notin E_2(R)$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e_{12}(1)\, e_{21}(-1)\, e_{12}(1)$$

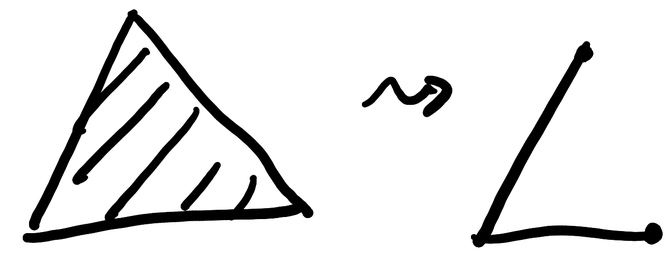
Origins of K_1

Whitehead's problem

When is a homotopy equivalence $f: X \to Y$ btw CW complexes simple (i.e. a composition of elementary expansions & collapses)?



Elementary expansion:
adjoin e_n & a free face e_{n-1}



Elementary collapse:
the reverse

Whitehead's thm:

f is simple $\iff$ $\tau(f) = 0$ in the Whitehead group

Whitehead torsion

To define Whitehead torsion :

$f: X \to Y$ homtopy equiv btw CW complexes

$\Rightarrow \tilde{f}: \tilde{X} \to \tilde{Y}$ equiv. btw covering space

$\Rightarrow \tilde{f}_*: C_*(\tilde{X}) \to C_*(\tilde{Y})$ chain homotopy equiv

<u>Obs.</u> $C_*(\tilde{X})$, $C_*(\tilde{Y})$ are free modules over $\mathbb{Z}[\pi_1]$, where $\pi_1 := \pi_1(X) = \pi_1(Y)$

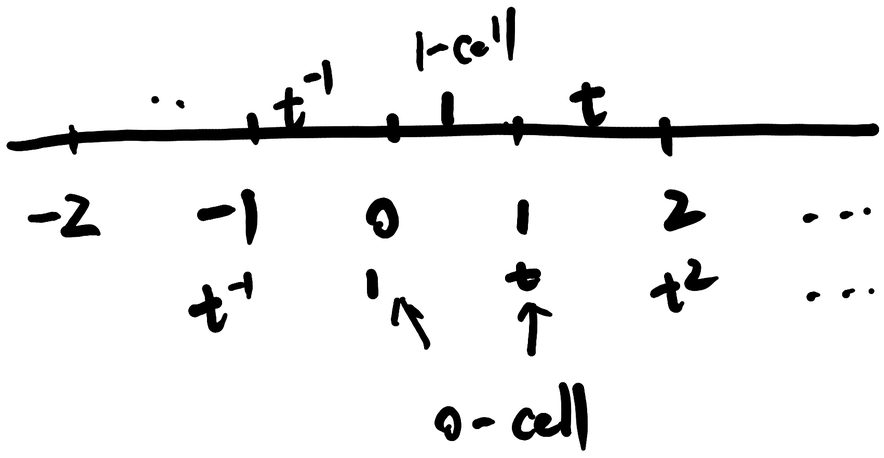
$C_n(\tilde{X}) = \mathbb{Z}[\pi_1]^{\#\, n\text{-cells in } X}$

e.g. $X = S^1$, $C_*(X)$: $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

1-cell 0-cell

$\tilde{X} = \mathbb{R}$, $C_*(\tilde{X}) =$ $0 \to \mathbb{Z}[t,t^{-1}] \xrightarrow{t-1} \mathbb{Z}[t,t^{-1}] \to 0$

0-cell



$1 \mapsto t-1$

$\pi_1 = \mathbb{Z}$

$\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$

$\tilde{f}_*: C_*(\tilde{X}) \to C_*(\tilde{Y})$ chain homotopy equiv

$\Rightarrow$ its cone $C_\cdot$ is contractible

i.e.

$\exists$ $s_k: C_k \to C_{k+1}$

$$\begin{array}{ccccccccccc}
0 \to & C_m & \to & C_{m-1} & \xrightarrow{\partial} & C_{m-2} & \to \cdots \to & C_1 & \to C_0 \to 0 \\
& \downarrow \mathrm{id} & \swarrow s_{m-1} & \downarrow & \swarrow s_{m-2} & \downarrow & & \downarrow & \swarrow s_0 \downarrow & \\
0 \to & C_m & \to & C_{m-1} & \to & C_{m-2} & \to \cdots \to & C_1 & \to C_0 \to 0
\end{array}$$

s.t. $s\partial + \partial s = \mathrm{id}$

$\partial s e_1 = e_1 \Rightarrow s e_1 = e_2 + e_2'$ $\quad \partial e_2' = 0$

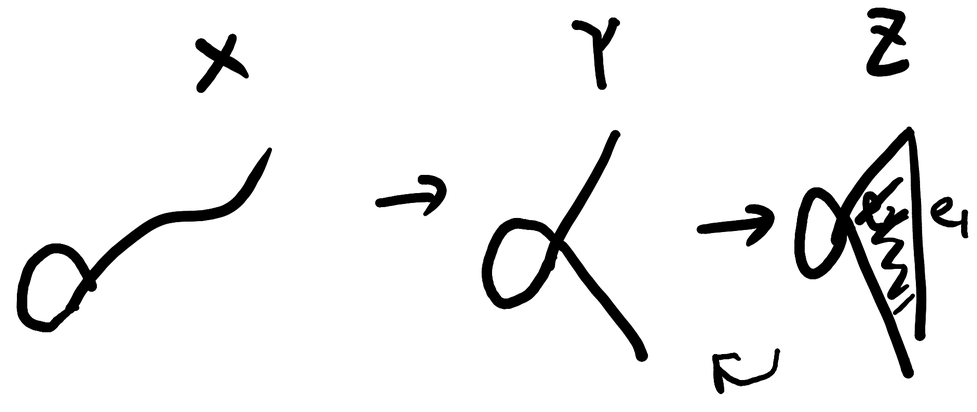
fact $\Rightarrow$ $\partial + s: C_{odd} \to C_{even}$ is an isom.

$$C_{odd} = \bigoplus_{k\,odd} C_k \oplus \mathbb{Z}[\pi_1] e_1 \qquad C_{even} = \bigoplus_{k\,even} C_k \oplus \mathbb{Z}[\pi_1] e_2$$

$\rightsquigarrow$ gives a matrix $T(f)$ in $GL_N(\mathbb{Z}[\pi_1])$

where $N = \mathrm{rk}\, C_{odd} = \mathrm{rk}\, C_{even}$

Ex/ $X \xrightarrow{f} Y \xrightarrow{\text{free pair}} Z$



$$\mathrm{Cone}(\tilde{f}_x) = 0 \to C_m \to \cdots \to C_2 \to C_1 \to C_0 \to 0$$

$$\mathrm{Cone}(\text{composition}) = 0 \to C_m \to \cdots \to C_2 \oplus \mathbb{Z}[\pi_1] \to C_1 \oplus \mathbb{Z}[\pi_1] \to C_0 \to 0$$

$$\Rightarrow \tau(\text{composition}) = \begin{pmatrix} \tau(f) & 0 \\ 0 & 1 \end{pmatrix}$$

$\rightsquigarrow$ natural place for $\tau(f)$ is $\varinjlim_N GL_N(\mathbb{Z}[\pi_1 X])$. $GL(\mathbb{Z}[\pi_1])$

Choosing different n-cells $\rightsquigarrow$ diff basis for $C_n \cong \mathbb{Z}[\pi_1]^{\#n\text{-cells}}$

$\Rightarrow$ look at $GL(\mathbb{Z}[\pi_1])/E(\mathbb{Z}[\pi_1])$

Geometry	Effect on $\tau(f)$	K-theory
add/contract a free pair	stabilization	$GL = \varinjlim GL_n$
choosing different n-cells	change basis $e_i \mapsto e_i + re_j$	$K_1 = GL/E$
reorder cells/ change lift to $\tilde{X}$	change basis $e_i \leftrightarrow e_j$	$Wh(X) = K_1(\mathbb{Z}[\pi_1])/\pm\pi_1$

Sep 30, 2026

Last time. Whitehead's group

$$Wh(X) = GL(\mathbb{Z}[\pi_1 X]) \,/\, \langle E(\mathbb{Z}(\pi_1 X)),\ \pm \pi_1 X \rangle$$

This provides the key ingredient for defining K_1:

History

1939 - 1950	Whitehead's work
1950s	Grothendieck introduced K_0
1959 - 1961	Atiyah - Hirzebruch introduced top'l K-theory
1962	Bass - Schanuel defined $K_1(R)$
1960s	Bass developed K_0, K_1 into systematic alg. K-theory

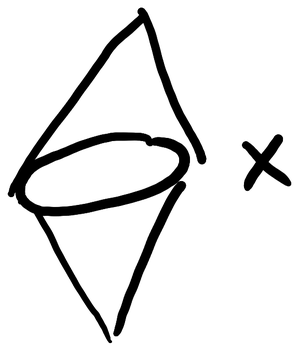
Landscape in early 1960s:

Topology	Algebra
vector bundles $K^0(X)$	proj. R-mod $K_0(R)$
stable autom $k'(X)$	??

↗

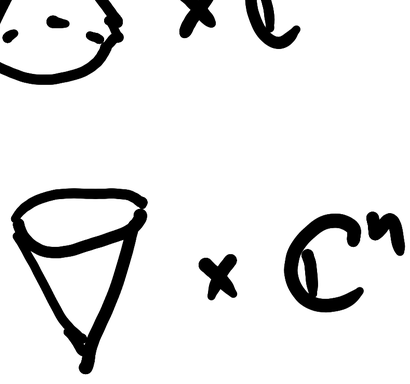
by Atiyah-Hirzebruch

$$k'(X) := \tilde{K}^0(\Sigma X)$$





↔ trivial v.b.  $\times \mathbb{C}^n$

+ v.b. on  $\times \mathbb{C}^n$

glued along X

↔ automorphisms of $X \times \mathbb{C}^n$

↑

Bass-Schanuel:

$$K_1(R) := GL(R)/E(R)$$

Recap. $K_1(R) = GL(R)/E(R)$

Whitehead's Lemma.

$$E(R) = [GL(R), GL(R)]$$

Thus $K_1(R) = GL(R)/[GL(R), GL(R)]$

is the abelianization of $GL(R)$

(For any gp G, $[G,G]$ is the commutator subgp of G generated by $ghg^{-1}h^{-1}$, $\forall g, h \in G$)

pf. ① $E(R) \subseteq [GL(R), GL(R)]$

because $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$

② Show $[A,B] = ABA^{-1}B^{-1} \in E(R)$:

$$[A,B] = \begin{pmatrix} A & \\ & A^{-1} \end{pmatrix} \begin{pmatrix} B & \\ & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & \\ & BA \end{pmatrix}$$

Exercise. $A \in GL_n(R) \Rightarrow \begin{pmatrix} A & \\ & A^{-1} \end{pmatrix} \in E_{2n}(R)$

Examples

- $K_1(F) = F^\times$, F field

  $A \mapsto \det A$

(all $A \in GL_n(F)$ can be row reduced to $\begin{pmatrix} \det A & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$)

e.g. $K_1(\mathbb{F}_q) = \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$

In general, we have a map $\det : K_1(R) \to R^\times$

Isom in many cases:

- $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}$

  Exercise. Show $K_1(R) = R^\times$ if R is Euclidean domain

- $K_1(k[x_1, \cdots, x_n]) = k^\times = K_1(k)$

  $\mathbb{A}^1$ - invariance. R regular $\Rightarrow$

  $K_1(R[x_1, \cdots, x_n]) \cong K_1(R)$

- $K_1(k[t,t^{-1}]) = k[t,t^{-1}]^\times = k^\times \oplus \mathbb{Z}$

$= K_1(k[t,t^{-1}]) \oplus K_0(k[t,t^{-1}])$

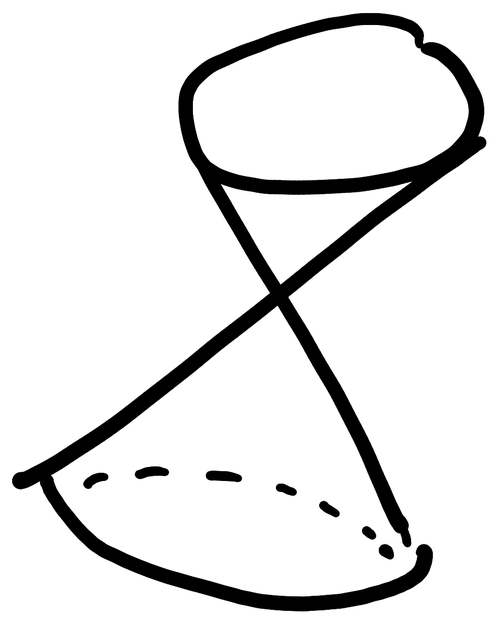
<u>Bass-Heller-Swan</u> : R regular

$\Rightarrow K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R)$

- $K_1\left(\mathbb{R}[x,y]/(x^2+y^2-1)\right) = \mathbb{R}^\times \oplus \mathbb{Z}/2\mathbb{Z}$

↑ generated by $A = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$

- $K_1\left(\mathbb{Q}[x,y,z]/(x^2+y^2-z^2)\right) = \mathbb{Q}^\times \oplus \mathbb{Q}$



Relation between K_0, K_1

C smooth curve

We have a map

$$k(C)^\times \xrightarrow{\text{div}} \bigoplus_{p \in C} \mathbb{Z}$$

$$f \longmapsto (\text{ord}_p f)_{p \in X}$$

$$\Rightarrow \quad K_1(k(C)) \longrightarrow \bigoplus_{p \in C} K_0(k(p))$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$k(C)^\times \xrightarrow{\text{div}} \bigoplus_{p \in C} \mathbb{Z}$$

In fact, this is the boundary map of the localization sequence:

Thm. R Dedekind domain

$F = \mathrm{Frac}(R)$

Then $\exists$ long exact seq. (localization seq.)

$$K_1(A) \to K_1(F) \xrightarrow{div} \bigoplus_{\mathfrak{p} \in \mathrm{Spec} A} K_0(k(\mathfrak{p}))$$

$$\| \qquad\qquad \|$$

$$A^\times \qquad F^\times$$

$$\to K_0(A) \to K_0(F) \to 0 .$$

Rmk. Efforts to extend seq. like these to the left motivates K_2 & higher K-theory

(Bass, Milnor, Quillen, ...)

Milnor's K_2

$$K_2(R) = \ker(St(R) \to E(R))$$

$$\|$$

$$\langle e_{ij}(r) : e_{ij}(\lambda) e_{ij}(\mu) = e_{ij}(\lambda+\mu)$$

$$[e_{ij}(\lambda), e_{jk}(\mu)] = e_{ik}(\lambda\mu) \quad i \neq k$$

$$[e_{ij}(\lambda), e_{kl}(\mu)] = 1 \quad i \neq l,\ j \neq k \rangle$$

$K_2(R)$ "=" nontrivial relations among elementary matrices in $E(R)$

"relations among relations"

Things I owe you

- K_1, K_2 for schemes (more general spaces?)
- Proof of localization seq., $\mathbb{A}^1$-inv.

  Bass-Heller-Swan
- Localization seq. beyond Dedekind domain?

  $\rightsquigarrow$ Gersten spectral seq.

(These are known classically, but we'll do them systematically through the framework of Quillen's higher K-theory)