The Grothendieck groups

Recall Riemann-Roch for curves:

Thm. C smooth proj curve, L l.b. on C

Then $\chi(C, L) = h^0(C, L) - h^1(C, L)$

$$= \deg L + 1 - g$$

Hirzebruch generalizes this to smooth proj. var. X and v.b. E on X:

$$\chi(X, E) \left(= \sum_i (-1)^i \dim H^i(X, E) \right)$$

$$= \int_X ch(E)\, td(T_X)$$

Ex (Curves). $ch(L) = 1 + c_1(L)$

$$td(T_X) = 1 + \frac{1}{2} c_1(T_C)$$

$$\Rightarrow \chi(C, L) = \int_C (1 + c_1(L))\left(1 + \tfrac{1}{2} c_1(T_C)\right)$$

$$= \int_C \left(c_1(L) + \tfrac{1}{2} c_1(T_C)\right)$$

$$= \deg L + \tfrac{1}{2} \deg T_C = \deg L + 1 - g$$

Ex (Surfaces) S surface. $L = \mathcal{O}_S(D)$

$$ch(L) = 1 + D + \tfrac{1}{2}D^2$$

$$td(T_S) = 1 + \tfrac{1}{2}c_1(T_S) + \tfrac{1}{12}\left(c_1(T_S)^2 + c_2(T_S)\right)$$

$$= 1 - \tfrac{1}{2}K_S + \tfrac{1}{12}(K_S^2 + c_2)$$

$$\Rightarrow \chi(S, L) = \int_S ch(L) \cdot td(T_S)$$

$$= \int_S \tfrac{1}{2}D^2 - \tfrac{1}{2}K_S.D + \tfrac{1}{12}(K_S^2 + c_2)$$

$$= \tfrac{1}{2}D.(D-K) + \tfrac{1}{12}(K_S^2 + c_2)$$

hol. Euler char

Exercise. Work out what HRR says for 3-folds

Grothendieck's viewpoint:

View HRR as a theorem about $p: X \to k$

$$\chi(X, E) = \int_X ch(E)\, td(T_X)$$

$$\sum_i (-1)^i \dim H^i(X, E) \qquad p_*\left(ch(E)\, td(T_X) \cap [X]\right)$$

$H^i(X,E) = R^i p_* E$

Need to make sense of $\sum_i (-1)^i R^i p_* E$

Introduce the <u>Grothendieck group of coherent sheaves</u>

$$G_0(X) = \mathbb{Z}\{[F] : F \in Coh(X)\} \Big/ [F] = [F'] + [F'']$$

whenever $0 \to F' \to F \to F'' \to 0$

χ is well-def on the quotient

For $f: X \to Y$ proper, define the <u>relative Euler char</u>

$$f_!: G_0(X) \to G_0(Y) \qquad [F] \mapsto \sum_i (-1)^i R^i f_* E$$

(Grothendieck group provides natural home for this)

One more problem: Need to make sense of $\underline{\underline{\otimes}}$ on G_0

a priori only for v.b.

Grothendieck group of v.b.:

$$K_0(X) = \mathbb{Z}\{[E] : E \in \mathrm{Vect}(X)\} \Big/ [E] = [E'] + [E'']$$

$$0 \to E' \to E \to E'' \to 0$$

Good news: Every coherent sheaf on smooth X is resolved by finitely many v.b. (Serre)

$$0 \to E_n \to \cdots \to E_2 \to E_1 \to \mathcal{F} \to 0$$

$$\Rightarrow [\mathcal{F}] = [E_1] - [E_2] + [E_3] - \cdots + (-1)^n [E_n] \in K_0(X)$$

$$\Rightarrow K_0(X) = G_0(X) \quad \text{when } X \text{ is smooth}$$

Rmk. This need not be true if X is singular

$\rightsquigarrow$ singularity K-theory

Grothendieck Riemann-Roch:

$f: X \to Y$ proper

$$\begin{array}{ccc} K_0(X) & \xrightarrow{f_!} & K_0(Y) \\ \downarrow \tau_X & & \downarrow \tau_Y \\ A^*(X)_{\mathbb{Q}} & \xrightarrow{f_*} & A^*(Y)_{\mathbb{Q}} \end{array}$$

where $\tau_X([E]) = ch(E)\, td(T_X) \cap [X]$

Reduction to HRR when $Y = \operatorname{Spec} k$ $\quad K_0(Y) = \mathbb{Z}$

$$f_*(\tau_X[E]) = \tau_Y(f_![E])$$

$$\| \qquad\qquad \| \quad \text{(}\tau_Y \text{ of } \textstyle\sum (-1)^i R^i p_* E\text{)}$$

$$\int_X ch(E)\, td(T_X) \qquad = ch\left(\textstyle\sum (-1)^i R^i p_* E\right) td(T_{pt})$$

$$= \sum (-1)^i \dim H^i(X, E)$$

K-theory for rings

$$K_0(R) := K_0(\mathrm{Spec}R) = \mathbb{Z}[\text{v.b. on } \mathrm{Spec}R]/\sim$$

$$\overset{\text{Swan's thm}}{=} \mathbb{Z}[\text{f.g. proj } R\text{-mod}] \Big/ [P\oplus Q] = [P]+[Q]$$

Ex/ $K_0(\text{field}) = \mathbb{Z}$, $K_0(\text{local ring}) = \mathbb{Z}$

$K_0(\text{PID}) = \mathbb{Z}$ (every f.g. proj mod / PID is free)

In general, we have $\mathrm{rk}: K_0(R) \to \mathbb{Z}$

$P \mapsto \mathrm{rk}P$

$$\rightsquigarrow K_0(R) = \mathbb{Z} \oplus \widetilde{K_0}(R)$$

(reduced k-theory)

$$K_0(k[x_1, \cdots, x_n]) = \mathbb{Z}$$

(Quillen-Suslin, also by $\mathbb{A}^1$-invariance)

Ex/ R Dedekind domain $\rightsquigarrow K_0(R) = \mathbb{Z} \oplus Cl(R)$

(by structure thm of DD)

e.g. $R = \mathbb{Z}[\sqrt{-5}]$, $K_0(R) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

$(2, 1+\sqrt{-5})$

Ex/ $K_0(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$ given by (rank, deg)

$K_0(C) = \mathbb{Z} \oplus \mathrm{Pic}(C)$

$E \mapsto (\mathrm{rk}\, E, \det E)$

Exercise. What is $K_0(\mathbb{P}^n)$?