Overview

$$H^k_{dR}(X) \xrightarrow[\sim]{\text{Hodge thm}} \{\text{harmonic } k\text{-forms}\}$$

$\|$ Hodge decomposition $\qquad\qquad$ $\{\|$ Kähler condition

$$\bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X) \xrightarrow[\sim]{\text{Hodge thm}} \bigoplus_{p+q=k} \{\text{harmonic } (p,q)\text{-forms}\}$$

$\|$ Dolbeault

$$\bigoplus_{p+q=k} H^q(X, \Omega^p_X)$$

Hodge decomposition

X compact Kähler manifold (e.g. smooth proj. var.)

$$\Rightarrow H^k_{dR}(X) \cong \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$$

Key input: the Hodge theorem

X Riemannian manifold, compact oriented, $\partial X = \phi$

Then $H^k_{dR}(X) := \{\text{closed } k\text{-form}\} / \{\text{exact } k\text{-form}\}$

$$\xrightarrow[\text{Hodge thm}]{\sim} \{\text{Harmonic } k\text{-form}\} =: \mathcal{H}^k$$

What are harmonic k-forms?

We have operators

$$d: \{k\text{-forms}\} \longrightarrow \{(k+1)\text{-forms}\}$$
$$d^*: \{(k+1)\text{-forms}\} \longrightarrow \{k\text{-forms}\}$$

d^* adjoint of d wrt the Riemannian metric

i.e. $\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle$

Useful formula: $\boxed{d^* = (-1)^{n(k+1)+1} * d *}$ on k-forms

where $*$ is the Hodge star operator

that sends $dx_i \mapsto dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

etc.

e.g. $X = S^1$, parametrized by θ

- d^*: 0-forms $\mapsto 0$
- On 1-forms $\alpha = f(\theta)\, d\theta$

$$d^*\alpha = -(*\,d\,*)(\alpha)$$
$$= - *d(f(\theta))$$
$$= - *(f'(\theta)\, d\theta)$$
$$= - f'(\theta)$$

$*1 = d\theta$

$*d\theta = 1$

e.g. X = torus $\mathbb{R}^2/(2\pi\mathbb{Z})^2$ with flat metric $dx^2 + dy^2$ oriented by $dx \wedge dy$

- d^*: 0-forms $\mapsto 0$
- On 1-forms $\alpha = P(x,y)\, dx + Q(x,y)\, dy$

$$d^*\alpha = -(P_x + Q_y)$$

- On 2-forms $\beta = F(x,y)\, dx \wedge dy$

$$d^*\beta = F_y\, dx - F_x\, dy$$

$\Delta = dd^* + d^*d$ is the Laplacian

(self-adj. & positive operator)

e.g. If locally the Riemannian metric

$$ds^2 = \sum_{i=1}^{n} dx_i \otimes dx_i$$

$$\Delta(f\, dx_{i_1} \wedge dx_{i_r}) = -\left(\sum_{j=1}^{m} \frac{\partial^2 f}{\partial x_j^2}\right) dx_{i_1} \wedge \cdots \wedge dx_{i_r}$$

We say a form w is <u>harmonic</u> if $\Delta w = 0$

$$\left(\iff dw = d^*w = 0 \right)$$

as $\langle \Delta w, w \rangle = \langle d^*w, d^*w \rangle + \langle dw, dw \rangle$

$$= \|dw\|^2 + \|dw\|^2$$

Thm (Orthogonal decomposition of forms)

For every k, there is an orthogonal decomposition

$$A^k(X, \mathbb{R}) = \mathcal{H}^k(X) \oplus dA^{k-1}(X, \mathbb{R}) \oplus d^*A^{k+1}(X, \mathbb{R})$$

differential k-forms — harmonic — exact — coexact

In particular, any closed form α can be written as $\alpha = h + d\beta$ for a unique choice of harmonic form h

- The existence of the orthogonal decomposition uses the theory of elliptic operator
- Decomposition of closed forms is an easy consequence:

$$\alpha = h + d\beta + d^*r \quad \text{closed}$$

harmonic

$$\Rightarrow d\alpha = dd^*r = 0$$

$$\Rightarrow \|d^*r\|^2 = \langle dd^*r, r\rangle = 0$$

$$\Rightarrow d^*r = 0$$

$$\Rightarrow \alpha = h + d\beta$$

This implies the Hodge theorem: For compact oriented Riemannian mfld with no boundary

$$H^k_{dR}(X) \xrightarrow{\sim} \mathcal{H}^k(X)$$

$$\alpha \longmapsto h$$

e.g. $X = S^1$, $H^1_{dR}(X) \cong \mathbb{R}\cdot d\theta$

- $\alpha = f(\theta)\,d\theta$ is <u>harmonic</u> $\Leftrightarrow$ $d\alpha = 0$, $d^*\alpha = f'(\theta) = 0$

$\Leftrightarrow f'(\theta) = 0$ i.e. f is const

- The decomposition of $\alpha = f(\theta)\,d\theta$ is given by

$$\alpha = \underbrace{\left(\int_0^1 f(t)\,dt\right) d\theta}_{\text{harmonic}} + \left(f(\theta) - \int_0^1 f(t)\,dt\right) d\theta$$

$$\text{!!}$$

$$g(\theta)$$

$$\int_0^1 g(\theta)\,d\theta = 0$$

$\Rightarrow$ g is the derivative of a periodic function

$\Rightarrow$ $g(\theta)\,d\theta$ is exact

e.g. $X = T^2 = \mathbb{R}^2 / (2\pi i\mathbb{Z})^2$

$H^1(X, \mathbb{R}) = \mathbb{R}^2$

- harmonic 1-forms $\quad$ (constant) $\cdot\, dx_1$

(constant) $\cdot\, dx_2$

- harmonic part of $\alpha = P(x,y)\,dx + Q(x,y)\,dy$

is $\quad h = \bar{P}\,dx + \bar{Q}\,dy$

where $\quad \bar{P}(x,y) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} P(x,y)\,dx\,dy$

$$\bar{Q}(x,y) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} Q(x,y)\,dx\,dy$$

Oct 30

Last time: Hodge theorem

X compact oriented Riemannian mfld with no boundary

$$\Rightarrow H^k_{dR}(X) \xrightarrow{\sim} \{\text{harmonic } k\text{-forms}\} =: \mathcal{H}^k$$

Today: Assume X is complex, and decompose $H^k_{dR}(X)$ into (p,q)-forms

Decomposition of harmonic forms

Further assume X is complex, with

complex dim n

real dim $2n$ real coordinates: $z_1, z_2, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n$

Then we have decomposition

$$A^k(X, \mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X)$$

smooth k-forms

$f \, dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_k}$ ← (p,q)-form if p dz_i and q $d\bar{z}_j$ are involved

f: complex-valued function

Want to show this induces decomp. at the level of coh.

Need to understand how d decomposes:

e.g. $d(f\,dz_1) = \underbrace{\frac{\partial f}{\partial z_2} \, dz_2 \wedge dz_1}_{(2,0)\text{-form } \partial(f\,dz_1)} + \underbrace{\frac{\partial f}{\partial z_1} \, d\bar{z}_1 \wedge dz_1 + \frac{\partial f}{\partial \bar{z}_2} \, d\bar{z}_2 \wedge dz_1}_{(1,1)\text{-form } \bar{\partial}(f\,dz_1)}$

Define $d = \partial + \bar{\partial}$

(p,q)-form $\xrightarrow{\partial}$ $(p+1, q)$-form

(p,q)-form $\xrightarrow{\bar{\partial}}$ $(p, q+1)$-form

∂ and $\bar{\partial}$ lead to cohomology theory just like d:

$$H^{p,q}_{\partial}(X) := \frac{\{\partial\text{-closed } (p,q)\text{-form}\}}{\{\partial\text{-exact } (p,q)\text{-form}\}}$$

Similar for $H^{p,q}_{\bar{\partial}}(X)$.

decomposes to (p,q)-forms

<u>Hope</u>: Just like $H^k_{dR}(X) \xrightarrow{\sim} \{\text{harmonic } \underline{k\text{-forms}}\}$,

we want $H^{p,q}_{\partial}(X) \xrightarrow{\sim} \{\partial\text{-harmonic } (p,q)\text{-forms}\}$

$$\Delta := \Delta_d := dd^* + d^*d$$

$$\Delta_\partial := \partial\partial^* + \partial^*\partial$$

$$\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$$

ω is Δ-<u>harmonic</u> if $\Delta\omega = 0$

Δ_∂-harmonic $\Delta_\partial\omega = 0$

$\Delta_{\bar{\partial}}$ .. .. $\Delta_{\bar{\partial}}\omega = 0$

This is where the Kähler conditions enter:

If X is a Kähler manifold, then

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$$

(Idea. If locally $ds^2 = \sum_{i=1}^n dz_i \otimes d\bar z_i$

$$\Delta_{\bar\partial}\left(f\, dz_I \wedge d\bar z_J\right)$$

$$= -2\sum_{i=1}^n \frac{\partial^2 f}{\partial z_i \partial \bar z_i}\, dz_I \wedge d\bar z_J$$

$$= -\frac{1}{2}\sum_{i=1}^n \left(\frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial y_i^2}\right) dz_I \wedge d\bar z_J$$

$$= \frac{1}{2}\Delta_d\left(f\, dz_I \wedge d\bar z_J\right)$$

The Kähler condition $\Leftrightarrow$

$$ds^2 = \sum_{i=1}^n dz_i \otimes d\bar z_i + O(\|z\|^2)\sum_{i,j} dz_i \wedge d\bar z_j\Big)$$

<u>Conclusion</u>. If X is compact Kähler, then

$$H^k_{dR}(X) \cong \{\text{harmonic } k\text{-forms}\}$$

$$\cong \bigoplus_{p+q=k} \{\text{harmonic } (p,q)\text{-forms}\}$$

$$\cong \bigoplus_{p+q=k} H^{p,q}_{\bar\partial}(X)$$

where $H^{p,q}(X) =$

$$\frac{\{\bar\partial\text{-closed } (p,q)\text{-forms}\}}{\{\bar\partial\text{-exact } (p,q)\text{-forms}\}}$$

Ex/ $X = \mathbb{C}/2\pi\mathbb{Z} + 2\pi i\mathbb{Z}$ complex torus

with coordinates $z = x + iy$ & standard flat metric

$$H^1_{dR}(X) \cong \mathbb{R}^2 \cong \underbrace{H^{1,0}(X)}_{\mathbb{R}\,dz} \oplus \underbrace{H^{0,1}(X)}_{\mathbb{R}\cdot d\bar{z}}$$

Consider $\alpha = (1+\cos x)\,dx \in H^1_{dR}(X)$

(closed; not exact bc $\int_0^{2\pi}(1+\cos x)\,dx = 2\pi$)

Can we decompose the 1-form α into $(0,1)$-form and $(1,0)$-form ?

$dx = \frac{1}{2}(dz + d\bar{z}) \Rightarrow$

$$\alpha = \underbrace{\frac{1+\cos x}{2}\,dz}_{(1,0)\text{-form}} + \underbrace{\frac{1+\cos x}{2}\,d\bar{z}}_{(0,1)\text{-form}}$$ NOT closed!

Instead, write $\alpha = dx + d(\sin x)$

harmonic representative ($d(dx)=0$ and $d^*(dx)=0$)

$$\Rightarrow dx = \underbrace{\frac{1}{2}\,dz}_{\in H^{1,0}(X)} + \underbrace{\frac{1}{2}\,d\bar{z}}_{\in H^{0,1}(X)}$$

Dolbeault's thm

$$H^{p,q}_{\bar\partial}(X) := \frac{\{\bar\partial\text{-closed } (p,q)\text{-forms}\}}{\{\bar\partial\text{-exact } (p,q)\text{-forms}\}}$$

$$= H^q(X, \Omega^p_X)$$

Sketch. The sheaf Ω^p_X is resolved by

$$0 \to \Omega^p_X \to A^{p,0}_X \xrightarrow{\bar\partial} A^{p,1}_X \xrightarrow{\bar\partial} \cdots \to A^{p,n}_X \to 0$$

smooth $(p,0)$-form

e.g. $\bar\partial\left(f(z,\bar z)\, dz_{i_1}\wedge\cdots\wedge dz_{i_p}\right)=0$

$\Leftrightarrow \frac{\partial f}{\partial \bar z_i}=0,\ \forall i \Leftrightarrow f$ is holomorphic

- exact by $\bar\partial$-Poincaré Lemma
- $A^{p,k}_X$ are acyclic by existence of partition of unity on C^∞ manifolds

$$\Rightarrow H^q(X, \Omega^p_X) = H^q\left(\cdots \to \Gamma(A^{p,q-1}_X) \xrightarrow{\bar\partial} \Gamma(A^{p,q}) \xrightarrow{\bar\partial} \Gamma(A^{p,q+1}) \to \cdots\right)$$

$$= \frac{\{\bar\partial\text{-closed } (p,q)\text{-forms}\}}{\{\bar\partial\text{-exact } (p,q)\text{-forms}\}}$$

$$= H^{p,q}_{\bar\partial}(X)$$

Similarly, the resolution

$$0 \to \underline{\mathbb{R}} \xrightarrow{d} A^0 \xrightarrow{d} A^1 \to A^2 \to \cdots \xrightarrow{d} A^n \to 0$$

- $\underline{\mathbb{R}}$: constant sheaf
- A^0: smooth functions on M
- A^1: sheaf of 1-form on M

$$\Rightarrow \quad H^k_{dR}(X) \cong H^k(X, \underline{\mathbb{R}}) \quad \text{sheaf coh}$$

$$= H^k(X, \mathbb{R}) \quad \text{singular coh}$$

$\uparrow$ the complex computing sing. coh can be viewed as a res of $\underline{\mathbb{R}}$

If M is a complex mfld, then

$$H^k(M, \mathbb{C}) \cong H^k_{dR}(M; \mathbb{C})$$

$$\| \qquad\qquad \wr \|$$

$$H^k(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^k_{dR}(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

Hodge decomposition

X compact Kähler mfld (e.g. smooth proj. var)

$$\Rightarrow \quad H^k_{sing}(X, \mathbb{C}) = H^k_{dR}(X) \otimes \mathbb{C}$$

$$\cong \bigoplus_{p+q=k} H^{p,q}(X)$$