Last time . X compact Kähler

(e.g. smooth proj var / $\mathbb{C}$)

$$\Rightarrow H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$$

Natural questions we can ask:

① What additional structure does $H^{p,q} = H^q(X, \Omega_X^p)$ have ? (Hard Lefschetz, Hodge-Riemann bilinear relations $\rightsquigarrow$ polarizable HS)

② Can we recover the geometry of X from its Hodge decomposition ?

(Period domains, period maps, Torelli thm)

③ How does the Hodge decomposition of X vary in smooth families (Variation of HS)

④ Algebraic subvarieties of X

$\rightsquigarrow$ class in $H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$

Does every class in $H^{p,p} \cap H^{2p}$ come from an alg. cycle ? (Hodge conjecture)

⑤ What if X is not compact, or not smooth?

(mixed Hodge structure)

⑥ What happens when a smooth family degenerates to a singular var? (limit mixed HS)

⑦ Can Hodge theory tell us about the topology of a map $\underbrace{f: X \to Y}$? (mixed Hodge modules)

family of var. that can have arbitrary sing

We will start with ⑤, and touch upon ④ and ⑦

Hodge to de Rham spectral sequence

X compact Kähler $\Rightarrow$ the spectral seq.

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1.

Construct the spectral sequence:

E_0 page: $E_0^{p,q} = A^{p,q}(X)$ (smooth (p,q)-forms on X)

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow & & \uparrow & & \\
A^{0,1}(X) & \to & A^{1,1}(X) & \to & \cdots \\
\bar\partial \uparrow & & \uparrow & & \\
A^{0,0}(X) & \xrightarrow{\partial} & A^{1,0}(X) & \to & A^{2,0}(X) \to \cdots
\end{array}$$

The total complex associated to this single complex

$$\begin{array}{ccccc}
A^{0,0}(X) & \xrightarrow{\partial+\bar\partial} & A^{1,0}(X) \oplus A^{0,1}(X) & \to & \cdots \\
\| & & \| & & \\
A^0(X) & \xrightarrow{d} & A^1(X) & \xrightarrow{d} & A^2(X)
\end{array}$$

is the de Rham complex computing $H^*(X, \mathbb{C})$

E_1 - page: By the acyclic resolution

$$\Omega_X^p \to A_X^{p,0} \to A_X^{p,1} \to \cdots ,$$

$H^*(X, \Omega_X^p)$ is computed by the complex

$$\Gamma(A_X^{p,0}) \longrightarrow \Gamma(A_X^{p,1}) \to \Gamma(A_X^{p,2}) \to \cdots$$

$$\| \qquad\qquad \|$$

$$A^{p,0}(X) \longrightarrow A^{p,1}(X) \to \cdots$$

Columns of the E_0 page

$\rightsquigarrow$ E_1 page:

$$H^2(X, \Theta_X) \to H^2(X, \Omega_X^1) \to \cdots$$

$$H^1(X, \Theta_X) \to H^1(X, \Omega_X^1) \to H^1(X, \Omega_X^2) \to \cdots$$

$$H^0(X, \Theta_X) \xrightarrow{\partial} H^0(X, \Omega_X^1) \xrightarrow{\partial} H^0(X, \Omega_X^2) \to \cdots$$

Claim: $\partial : H^q(X, \Omega_X^p) \longrightarrow H^q(X, \Omega_X^{p+1})$ is the zero map

$(H^q(X, \Omega_X^p) \cong \{$harmonic (p,q)-forms$\}$, so $\partial \omega = 0\}$

since ω is $\Delta_{\bar\partial}$-harmonic

Thus the Hodge to de Rham spectral seq

$$E_1^{p,q} = H^q(X, \Omega_X^p) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates

(Alternatively, Hodge decomposition + dimension count

$\Rightarrow$ degeneration

But degeneration $\not\Rightarrow$ decomposition)

Note. This fails for singular var!

Ex/ $C = \{Y^2Z = X^3\} \subseteq \mathbb{P}^2_{\mathbb{C}}$

Exercise. Use the open cover $U = \operatorname{Spec} \mathbb{C}[t^2, t^3]$

$V = \operatorname{Spec} \mathbb{C}[t^{-1}]$

to compute the Čech coh

$$H^1(C, \mathcal{O}_C) \cong \mathbb{C}[t, t^{-1}] \Big/ \mathbb{C}[t^2, t^3] + \mathbb{C}[t^{-1}] \cong \mathbb{C} \cdot t$$

$$H^1(C, \Omega^1_C) \cong \mathbb{C}[t, t^{-1}]\,dt \Big/ t\mathbb{C}[t]\,dt + t^{-2}\mathbb{C}[t^{-1}]\,dt$$

$$\cong \mathbb{C} \cdot dt \oplus \mathbb{C} \cdot t^{-1}\,dt$$

The E_1-differential

$$H^1(C, \mathcal{O}_C) \to H^1(C, \Omega^1_C)$$

$$t \mapsto dt$$

is nonzero!

will need to replace Ω_X^p by the Du Bois complex $\underline{\Omega}_X^p$

Hodge filtration

On differential forms, we have a natural filtration

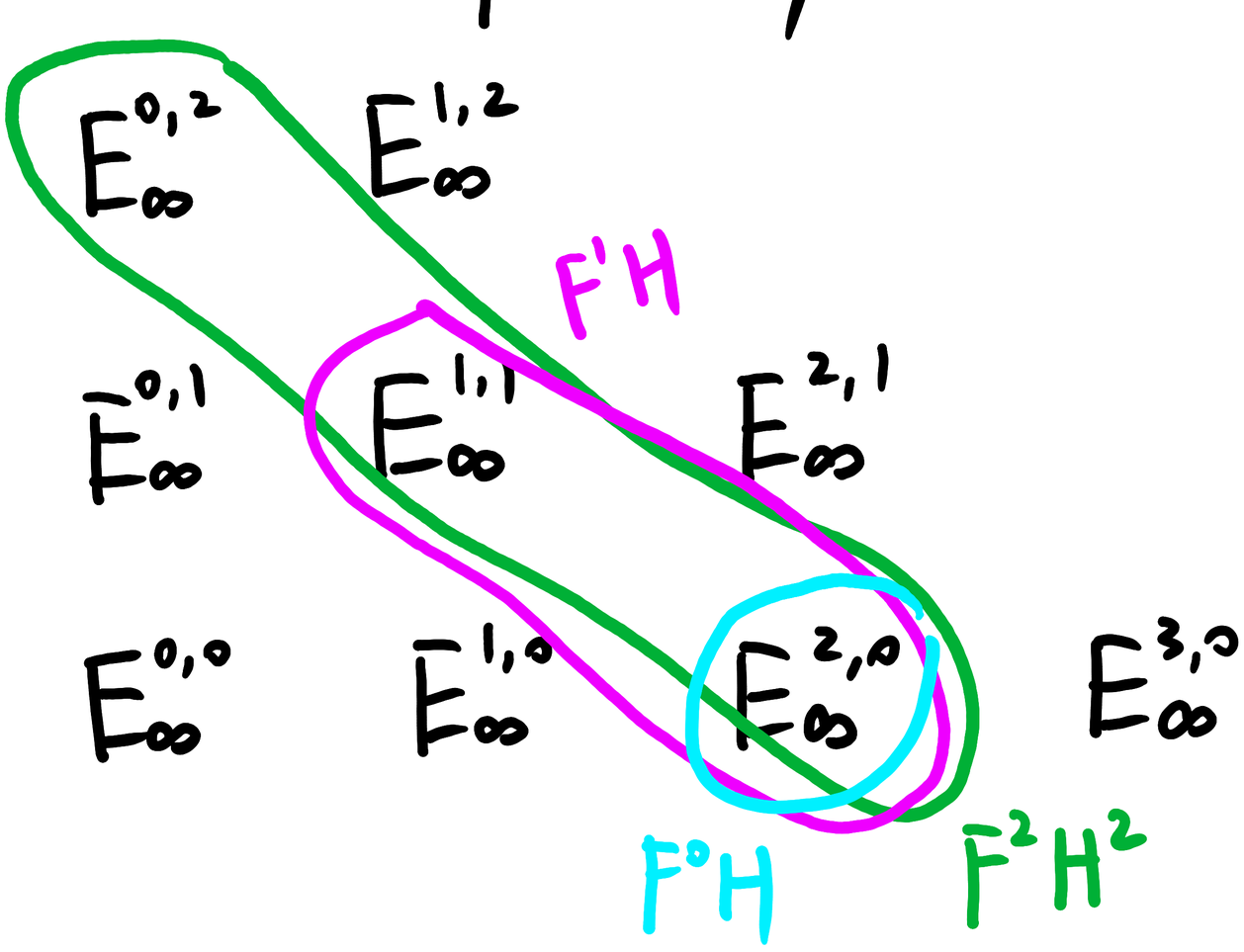
$$F^p A^*(X) = \bigoplus_{\alpha \geq p} A^{\alpha,*}(X)$$

This induces a filtration on cohomology

$$F^p H^k(X, \mathbb{C}) = \operatorname{im}\left(H^k(F^p A^*(X)) \to H^k_{dR}(X) \right)$$

$$\| \qquad\qquad H^k(X, \mathbb{C})$$

General theory of spectral seq tells us this filtration is computed by the E_∞ - page:



$$H^2(X, \mathbb{C}) = F^2 H^2 \supseteq F^1 H^2 \supseteq F^0 H^2 \supseteq F^{-1} H^2 = 0$$

$$E_\infty^{2,0} = Gr_F^0 H^2 \qquad E_\infty^{1,1} = Gr_F^1 H^2 \qquad E_\infty^{0,2} = Gr_F^2 H^2$$

More generally,

general theory of spectral seq

$$Gr_F^p H^k(X, \mathbb{C}) \cong E_\infty^{p,k-p}$$

$$= E_1^{p,k-p} = H^{k-p}(X, \Omega_X^p)$$

degeneration at E_1 page

Takeaway: $Gr_F^p H^k(X, \mathbb{C}) \cong H^{k-p}(X, \Omega_X^p)$