Gersten's conjecture

Recall for a Dedekind domain R with $F = \operatorname{Spec} R$, we have LES

$$\cdots \to K_2(F) \to \bigoplus_{\substack{p \neq 0 \\ \max}} K_1(k(p)) \to K_1(R) \to K_1(F) \to \bigoplus_{\substack{p \neq 0 \\ \text{max ideal} \\ \text{in } R}} K_0(k(p)) \to K_0(R) \to K_0(F) \to 0$$

This follows from the localization thm, applied to

$$B = \text{f.g. torsion } R\text{-mod} \cong \bigoplus_{\substack{m \\ \text{maximal}}} \text{f.g. } R\text{-mod supp on } \{m\}$$

$$\rightsquigarrow K_i(B) \overset{\text{devissage}}{\cong} \bigoplus_m G_i(R/m) \overset{R/m \text{ is regular}}{\cong} \bigoplus_m K_i(R/m)$$

$$A = \text{f.g. } R\text{-mod} \rightsquigarrow K_i(A) \cong G_i(R) \overset{R \text{ is regular}}{\cong} K_i(R)$$

$$A/B = \text{f.g. } F\text{-mod} \rightsquigarrow K_i(A/B) \cong K_i(F)$$

To extend this to higher dim, we filter A by dim supp of the R-mod: $\rightsquigarrow$ Gersten spectral seq.

R = commutative Noetherian ring, $\dim R < \infty$

$M(R) :=$ all f.g. R-mod

$M^i(R) :=$ f.g. R-mod with codim supp $\geq i$

(so we have filtration $d = \dim R < \infty$

$M(R) = M^0(R) \supseteq M^1(R) \supseteq \cdots \supseteq M^d(R) \supseteq 0$)

Fact: • Each $M^i(R)$ is a Serre subcat of $M(R)$

(i.e. closed under subobj & quotient,

so $M^i(R)/M^{i+1}(R)$ makes sense)

• $M^i(R)/M^{i+1}(R) \cong \bigoplus_{\mathrm{ht}\,p = i}$ f.g. R-mod supp at p

Devissage $\Rightarrow$ $K_i\left(M^i(R)/M^{i+1}(R)\right) \cong \bigoplus_{\mathrm{ht}\,p=i} G_i(k(p))$

$\cong \bigoplus_{\mathrm{ht}\,p=i} K_i(k(p))$

Localization $\Rightarrow$

$$\cdots \to \bigoplus_{ht p = i} K_{n+1}(k(p)) \xrightarrow{j} K_{n+1}(M^{i+1}(R)) \xrightarrow{i} K_{n+1}(M^i(R))$$

$$\xrightarrow{\partial} \bigoplus_{ht p = i} K_n(k(p)) \to \cdots \qquad \text{for each } i$$

Patching these together gives a spectral seq

$$E_1^{p,q} = \bigoplus_{ht p = p} K_{-p-q}(k(p)) \Rightarrow G_{-p-q}(R)$$

by general machinery of exact couples

The E_1 differentials are given by

$$\bigoplus_{ht p = p} K_{-p-q}(k(p)) \xrightarrow{j} K_{-p-q}(M^{p+1}(R))$$

$$\xrightarrow{\partial} \bigoplus_{ht p = p+1} K_{-p-q-1}(k(p))$$

Converges to "what we are filtering", namely

$$K_i(M(R)) = G_i(R)$$

E_1 page :

For simplicity, suppose $\operatorname{Spec} R$ is irred,

so $\bigoplus_{htp=0} K_i(k(p)) = K_i(F)$

where $F = \operatorname{frac}(R)$

	$p=0$	$p=1$		
$q=0$	$K_0(F)$	0	0	0
$q=-1$	$K_1(F) \to \bigoplus_{htp=1} K_0(k(p))$	0	0	0
	$K_2(F) \to \bigoplus_{htp=1} K_1(k(p)) \to \bigoplus_{htp=2} K_0(k(p))$			0
	$K_3(F) \to \cdots$			

entries above the diagonal are zero, since there is no negative K-theory

The $(-q)$-th row of E_1 looks like

$$K_q(F) \to \bigoplus_{htp=1} K_{q-1}(k(p)) \to \bigoplus_{htp=2} K_{q-2}(k(p)) \to \cdots \qquad (*)_q$$

Gersten's Conj

R regular local ring $\Rightarrow (*)_q$ is a resolution of $K_q(R)$

e.g. $q=0$ $\quad K_0(R) \xrightarrow{\simeq} K_0(F)$ is isom

$\| \qquad\qquad \|$

$\mathbb{Z} \qquad\qquad \mathbb{Z}$

(R regular $\Rightarrow$ integral domain)

$q=1$ we have exact seq

$$0 \to K_1(R) \to K_1(F) \to \bigoplus_{ht\,p=1} K_0(k(p)) \to 0$$

$\|$ R local ring $\quad \|$ $\qquad\qquad \|$

$$0 \to R^\times \to F^\times \xrightarrow{div} \bigoplus_{ht\,p=1} \mathbb{Z} \to 0$$

$$f \longmapsto \operatorname{ord}_p f$$

exact here since R is normal (so $R = \bigcap_{ht\,p=1} R_p$)

exact since R is UFD (so we can choose uniformizers π_p with $\operatorname{ord}_q(\pi_p) = \begin{cases} 1, & q=p \\ 0, & q \neq p \end{cases}$)

[$\operatorname{coker}(div) = Cl(R) = 0$ as R is UFD]

Thm (Quillen)

Gersten's conj. is true when R is the localization of a regular finite type k-algebra.

(e.g. local rings of alg. varieties)

Idea. ① Suffices to show the maps

$$K_n(M^{i+1}(R)) \longrightarrow K_n(M^i(R)) \text{ are zero}$$

(bc the LES from localization thm splits into SES, which can be assembled into the resolution of $(*)_q$)

② (Reduction by general hyperplanes)

Since $K_n(M^{i+1}(R)) = \varinjlim_{t \text{ NZD}} K_n(M^i(R/tR))$,

it suffices to show the maps

$$K_n(M^i(R/tR)) \to K_n(M^i(R)) \text{ are zero}$$

③ i) If R looks like $(R/tR)[t]$,

then additivity thm shows the map is zero

↓

as in the pf of fundamental thm of G-theory:

$M \mapsto M$ $\quad G_i(R) \to G_i(R[t])$

is zero

ii) In general, reduce to cases like this

using a "variant of Noether normalization"

(more details in Quillen / Weibel's books)

Rmk. Status of Gersten's conjecture

- Equicharacteristic [by Panin, building on Quillen, Popescu, Grothendieck]

  ($\mathrm{char}(R) = \mathrm{char}(R/m)$

  e.g. $\mathbb{C}[[x,y]]$, $\mathbb{Q}[[x,y]]$, $\mathbb{F}_p[[x,y]]$)

known by

- Mixed characteristic $(0,p)$

  ($\mathrm{char}(R) = 0$, $\mathrm{char}(R/m) = p$

  e.g. $\mathbb{Z}_{(p)}$, $\mathbb{Z}_p$, $\mathbb{Z}_p[[x,y]]$)

Many work on this, recently claimed by OpenAI: ① $\exists$ 2-dim regular local A of mixed char $(0,5)$ s.t.

$K_5(A) \to K_5(\mathrm{Frac}\, A)$ is not inj.

② $\exists$ 2-dim ...

$K_3(A) \to K_3(\mathrm{Frac}\, A)$ is not inj.

X Noetherian scheme.

Define the K-theory sheaf $\mathcal{K}_q$ on X as the sheafification of $U \mapsto K_q(U)$

There is a complex of sheaves

$$0 \to \mathcal{K}_q \to \bigoplus_{\operatorname{codim} x = 0} (i_x)_* K_q(k(x))$$

$$\to \bigoplus_{\operatorname{codim} x = 1} (i_x)_* K_{q-1}(k(x))$$

$$\to \cdots$$

$$\to \bigoplus_{\operatorname{codim} x = q} (i_x)_* K_0(k(x)) \to 0 \qquad (\bigstar)$$

At each stalk at $x \in X$, this reduces to $(*)_q$ for $R = \mathcal{O}_{X,x}$

(we are using that K-theory commutes w/ colimits)

Cor. $(\bigstar)$ is exact at smooth points of X

When X is smooth, we get <u>Bloch's formula</u>:

$\Downarrow$

$(\star)$ is an acyclic resolution of K_q

$$H^p(X, K_p) = \operatorname{Coker}\Big(\bigoplus_{\mathrm{ht}\,p = p-1} K_1(k(p)) \to \bigoplus_{\mathrm{ht}\,p = p} K_0(k(p)) \Big)$$

$$\| $$

$$\bigoplus_{\mathrm{ht}\,p = p-1} k(p)^\times \to \bigoplus_{\mathrm{ht}\,p = p} \mathbb{Z}$$

$$= CH^p(X)$$

(the Chow group of codim p cycles)

e.g. $p = 1$, we get the

$$CH^1(X) \cong Cl(X) \cong \mathrm{Pic}(X)$$