So far we've seen

$$K_0(R) = \mathbb{Z}\{\text{proj. } R\text{-mod}\}/\,[P\oplus Q] - [P]\oplus[Q]$$

$$K_1(R) = GL(R)/E(R)$$

$$K_2(R) = \ker\big(St(R) \to E(R)\big) = H_2(E(R), \mathbb{Z})$$

↑ general theory of universal central extensions

Quillen's idea

Define $K_i(R) := \pi_i$(a space constructed from R) for $i \geq 1$

How to construct this space ?

$R \rightsquigarrow GL(R) \rightsquigarrow$ classifying space $BGL(R)$

$BGL(R)^+$ ← (+-construction) ← makes $\pi_1 = K_1$, $\pi_2 = K_2$, leaves H_* unchanged

$K_i(R) \xleftarrow{\pi_i} BGL(R)^+$

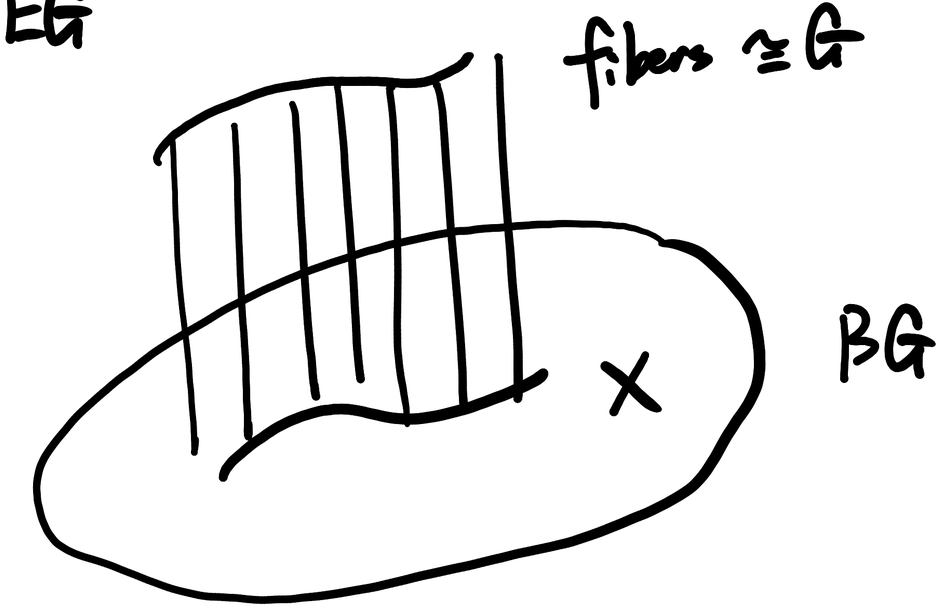
for $i \geq 1$

Classifying space

For a top'l group G, the classifying space BG "classifies all principal G-bundles", i.e. $\exists$ bijection

$$\left\{\begin{array}{c}\text{maps } X \to BG \\ \text{up to homotopy}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{principal } G\text{-bundles} \\ \text{over } X \text{ up to Iso}\end{array}\right\}$$

$$f \longleftrightarrow f^* EG$$



There is a "universal principal G-bundle" $EG \longrightarrow BG$ with EG contractible

Construction of EG and BG

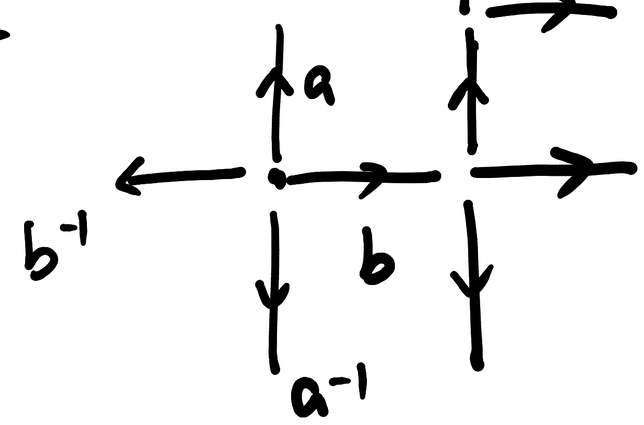
- Take EG = any contractible Hausdorff space on which G acts freely & properly discontinuously
- $BG := EG/G$

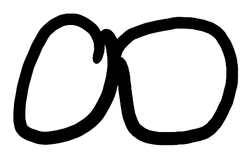
How to construct EG ?

① By inspection

e.g. • $G = \mathbb{Z}$, $EG = \mathbb{R}$, $BG = EG/G = S^1$

• G = free group on 2 generators

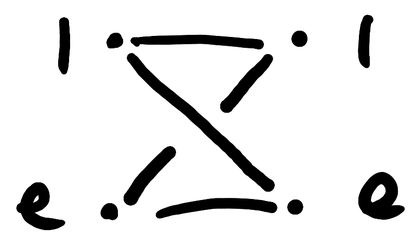
$EG =$ 

$BG =$ 

• $G = \mathbb{Z}/2\mathbb{Z}$ $\quad EG = ?$

② (Eilenberg - Maclane)

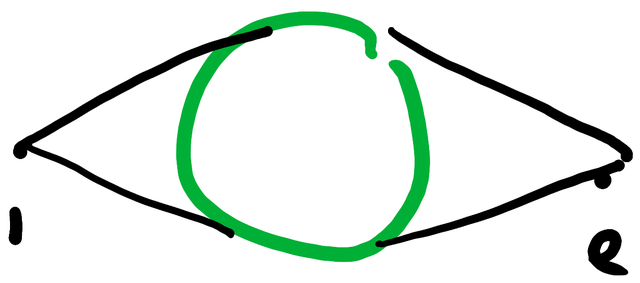
Take joins : $EG := G * G * \cdots$

e.g. $G = \mathbb{Z}/2\mathbb{Z}$ $\quad G * G =$  $\simeq S^1$

1 •

e •

$G * G * G$



$\simeq \Sigma S^1 \simeq S^2$

$\Rightarrow EG = S^\infty$

$BG = S^\infty/G \simeq \mathbb{RP}^\infty$

e.g. $G = \mathbb{Z}$ $\rightsquigarrow EG = G * G * \cdots$ is a more complicated model (homotopic to S^1)

This construction shows EG (hence BG) can be taken to be a CW complex

③ Use <u>simplicial sets</u>

↑

diagrams of the form

$$\cdots \; X_3 \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; X_2 \;\substack{\leftarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow\\ \leftarrow}\; X_1 \;\substack{\rightarrow\\ \rightarrow\\ \leftarrow}\; X_0$$

e.g. G is the simplicial set (top'l group)

$$G: \quad \cdots \rightarrow G_2 \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; G_1 \;\substack{\rightarrow\\ \rightarrow\\ \leftarrow}\; G_0$$

$$\begin{array}{ccc} \| & \| & \| \\ G^3 & G^2 & G \end{array}$$

$(g_0, g_1) \mapsto g_0$

$\mapsto g_1$

$(g_0, g_1) \leftarrow g$

$(g_0, g_1, g_2) \mapsto (g_0, g_2)$

(g_0, g_2)

(g_0, g_1)

e.g. For a category $\mathcal{C}$, the <u>nerve</u> of $\mathcal{C}$ is

$$N_{erv}(\mathcal{C}): \cdots \to N_2 \rightrightarrows N_1 \rightrightarrows N_0$$

N_2 = {composable morphisms} $= \{X \to Y \to Z\}$

$N_1 = Mor(\mathcal{C})$ $\quad (X \to Y) \mapsto X$, $\mapsto Y$

$N_0 = Obj(\mathcal{C})$

This is a simplicial set, in fact an infinity category

e.g. View G as a category where

- obj = $\{\cdot\}$
- morphism = G



Exercise. Show that $N_{erv}(G)$ is

$$\cdots \rightrightarrows G \times G \underset{\sigma_i}{\overset{d_i}{\rightrightarrows}} G \rightrightarrows .$$

We can take the <u>geometric realization</u> $|X_\cdot|$ of a simplicial set $X_\cdot$.

$$|X_\cdot| = \left(\coprod_{n \geq 0} X_n \times \Delta^n \right) / \sim$$

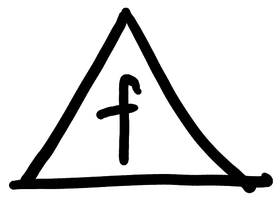
where $(d_i x, t) \sim (x, \delta_i t)$

$(s_i x, t) \sim (x, \sigma_i t)$

i.e. glue all of $\cdot v$ for $v \in X_0$



for $e \in X_1$



for $f \in X_2$ etc.

along maps specified by

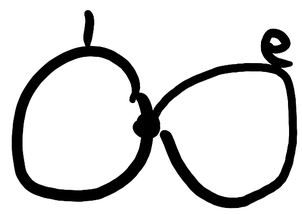
e.g. For $G = \mathbb{Z}/2\mathbb{Z}$, we need to glue







etc.

Exercise. $|Nerv(G)| =$ 

Defn. The classifying space of a category $\mathcal{C}$ is

$$B\mathcal{C} := |Nerv(\mathcal{C})|$$

Rmk. If G is a top'l group, with product top'l induced from G

$$|Nerv(G)| = \left(\coprod_{n \geq 0} G^n \times \Delta^n \right) / \sim$$

gives BG

Homotopy and homology groups of BG

① $\pi_{k+1}(BG) = \pi_k(G)$ for $k \geq 0$

When G is discrete, $\pi_k(G) = \begin{cases} G, & k=1 \\ 0, & k \geq 2 \end{cases}$

Exercise. Prove this using the LES in π_k induced by the fibration

$$\begin{array}{ccc} G & \to & EG = * \\ & & \downarrow \\ & & BG \end{array}$$

② $H_*(BG; \mathbb{Z}) = H_*(G; \mathbb{Z})$ (group homology)

Proof idea.

Cellular complex on EG is a free $\mathbb{Z}[G]$-mod;
taking $/G$ corresp. to $\otimes_{\mathbb{Z}G} \mathbb{Z}$;
the resulting complex computes the group homology

	$K_i(R)$	$\pi_i(BGL(R))$ ← as a discrete gp
$i=0$	$K_0(R)$	0
$i=1$	$GL(R)/E(R)$	$GL(R)$
$i=2$	$H_2(E(R),\mathbb{Z})$	0
$i=3$		0
		⋮

Quillen's plus construction

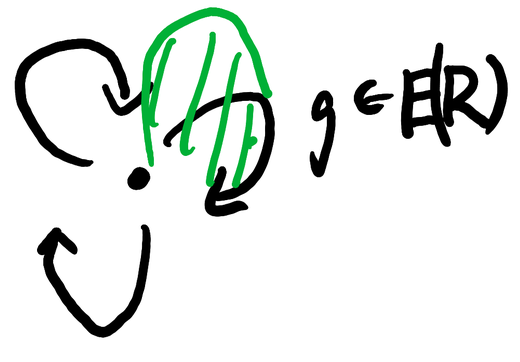
Goal: Construct $BGL(R)^+$ s.t. $\pi_i(BGL(R)^+) = K_i(R)$ for $i \geq 1$

- Attach 2-cells for each $g \in E(R)$:

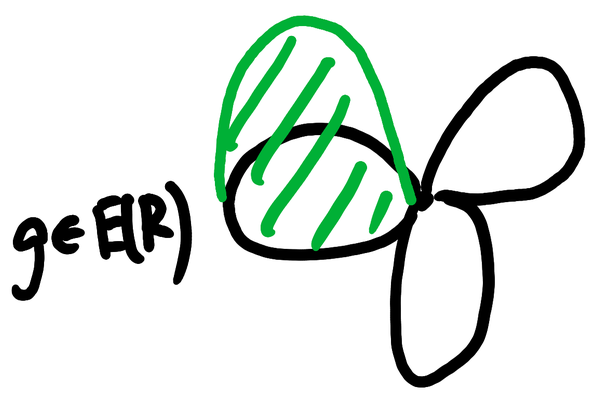
  $E(R) \subseteq \pi_1(BGL(R)) = GL(R)$, so each elem in $E(R)$ corresponds to a loop

  $g_i : \partial B^2 = S^1 \longrightarrow BGL(R)$

Let Y be the CW complex obtained by attaching 2-cells e_i^2 to X



- Note the attached e_i^2 is nonzero in $H_2(\Gamma)$



$\leftarrow$  is a 2-chain

- It is 2-cycle because the boundary $g = g_1^{-1}g_2^{-1}g_1g_2$
- It is not a boundary

To see this mathematically,

consider the relative LES in homology:

$$H_2(\Gamma) \to H_2(\Gamma, BGL(R)) \xrightarrow{\partial} H_1(BGL(R))$$

$$= H_2(\Gamma/BGL(R)) \qquad \text{Hurewicz}$$

$$= \bigoplus_i \mathbb{Z}[e_i^2] \qquad \pi_1(BGL(R))_{ab}$$

$\partial e_i^2 = 0$ (if e_i^2 corresp. to $g_i \in E(R) = [E(R), E(R)]$,

$\partial e_i^2 = a^{-1}b^{-1}ab$ $\quad = \vec{a}\vec{b}^{-1}ab$

$= 0$ in $(\pi_1)_{ab}$)

$\Rightarrow$ $H_2(\Gamma) \twoheadrightarrow H_2(\Gamma, BGL(R))$

$\Rightarrow$ Each e_i^2 is nonzero in $H_2(\Gamma)$

- Attach 3-cells for each $e_i^2 \in H_2(\Gamma)$ to get $BGL(R)^+$:

$$\cdots \to H_3(BGL(R)^+) \to H_3(BGL(R)^+, \Gamma) \xrightarrow{\partial} H_2(\Gamma) \to \cdots$$

$$e_i^3 \longmapsto e_i^2$$

This corrects H_2, and does not change $H_3, H_4, \cdots$

One can carry out this construction more generally:

Thm (Quillen)

Let X be a connected CW complex with base point x_0 (chosen from the 0-skeleton)

$\pi \in \pi_1(X, x_0)$ perfect normal subgroup

(i.e. $[\pi, \pi] = \pi$)

Then there exists a new CW complex $X \to X^+$ (unique up to homotopy) s.t.

1) $\pi_1(X, x_0) \longrightarrow \pi_1(X^+, x_0)$

2) $H_*(X^+, X; M) = 0$ for any π_1/π-module M

Rmk. In our case, $X = BGL(R)$

$\pi = E(R)$

Resulting $X^+ = BGL(R)^+$ satisfies

$\pi_i(BGL(R)^+) = K_i(R)$ for $i = 1, 2$

One final problem:

$BGL(R)^+$ is path-connected

$\Rightarrow \pi_0(BGL(R)^+) = 0$ does not give $K_0(R)$

← with discrete topology

Define $K(R) := BGL(R)^+ \times K_0(R)$, $i \geq 0$

$K_i(R) := \pi_i(K(R))$ (agrees w/ classical def of K_0, K_1, K_2)

↑

computed relative to a base point in $BGL(R)^+ \times \{0\}$

This is a great construction, but

- hard to prove thms with it
- the $\times K_0(R)$ is a bit unnatural
- works only for rings. does not generalize to categories

$\rightsquigarrow$ Q-construction